Smith Chart Derivation

V1. September 15, 2013
V2. October 28, 2014


Phillip H. Smith (1905 -1987)

[Update]2014-10-28: Thanks for **Tom West** (tomwest37@gmail.com)
the small mistake about the center of the real circle is corrected.

Consider the reflection coefficient (lossless case)

$$\Gamma(z) = \frac{V_0^-}{V_0^+} e^{j2\beta z} = |\Gamma| e^{j2\beta z}$$

It is a complex number

$$\Gamma = |\Gamma|\angle\theta = \Gamma_R + j\Gamma_I$$

Recall that Γ can be expressed as Z_L , Z_0

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Define *normalized load impedance* z_L

$$z_L = \frac{Z_L}{Z_0}$$

Thus

$$\Gamma = \frac{z_L - 1}{z_L + 1}$$

Express $z_L$as Γ

$$z_L = \frac{1+\Gamma}{1-\Gamma} = \frac{1+\Gamma_R + j\Gamma_I}{1-\Gamma_R - j\Gamma_I}$$

Rationalize

$$z_L = \frac{1-\Gamma_R^2 - \Gamma_I^2 + j2\Gamma_I}{(1-\Gamma_R)^2 + \Gamma_I^2}$$

The normalized load impedance is also a complex number

$$z_L = z_R + jz_I$$

The z_R is the normalized load resistance, while z_I is the normalized load reactance, thus

$$z_R = \frac{1-\Gamma_R^2 - \Gamma_I^2}{(1-\Gamma_R)^2 + \Gamma_I^2} \qquad z_I = \frac{2\Gamma_I}{(1-\Gamma_R)^2 + \Gamma_I^2}$$

This two equations are circle equations. The following will illustrate how to obtain the 2 circle equations

$$z_R = \frac{1-\Gamma_R^2 - \Gamma_I^2}{(1-\Gamma_R)^2 + \Gamma_I^2}$$

$$\iff \quad z_R\left[(1-\Gamma_R)^2 + \Gamma_I^2\right] = 1 - \Gamma_R^2 - \Gamma_I^2$$

$$\iff \quad z_R - 2z_R\Gamma_R + z_R\Gamma_R^2 + z_R\Gamma_I^2 = 1 - \Gamma_R^2 - \Gamma_I^2$$

$$\iff z_R - 1 - 2z_R\Gamma_R + (1+z_R)\Gamma_R^2 + (1+z_R)\Gamma_I^2 = 0$$

$$\iff \frac{z_R - 1}{1+z_R} - \frac{2z_R\Gamma_R}{1+z_R} + \Gamma_R^2 + \Gamma_I^2 = 0$$

$$\iff \underbrace{\frac{z_R - 1}{1+z_R} - \overbrace{\left(\frac{z_R}{1+z_R}\right)^2}^{\text{Completing square}}}_{\text{Become}} \underbrace{+ \overbrace{\left(\frac{z_R}{1+z_R}\right)^2}^{} - \frac{2z_R\Gamma_R}{1+z_R} + \Gamma_R^2}_{\text{Become}} + \Gamma_I^2 = 0$$

$$\iff \frac{-1}{(1+z_R)^2} \quad + \quad \left(\Gamma_R - \frac{z_R}{1+z_R}\right)^2 \quad + \Gamma_I^2 = 0$$

$$\iff \left(\Gamma_R - \frac{z_R}{1+z_R}\right)^2 + \Gamma_I^2 = \left(\frac{1}{1+z_R}\right)^2$$

$$\boxed{\text{Circle with center } \left(\frac{z_R}{1+z_R}, 0\right) \text{ , radius } \frac{1}{1+z_R}}$$

$$z_I = \frac{2\Gamma_I}{(1-\Gamma_R)^2 + \Gamma_I^2}$$

$$\iff \quad (1-\Gamma_R)^2 + \Gamma_I^2 = \frac{2\Gamma_I}{z_I}$$

$$\iff \quad (\Gamma_R - 1)^2 + \Gamma_I^2 - \frac{2\Gamma_I}{z_I} + \quad \underbrace{\frac{1}{z_I^2} - \frac{1}{z_I^2}}_{\text{Completing square}} \quad = 0$$

$$\iff \quad (1-\Gamma_R)^2 + \left(\Gamma_I - \frac{1}{z_I}\right)^2 = \frac{1}{z_I^2}$$

$$\boxed{\text{Circle with center } \left(1, \frac{1}{z_I}\right) \text{ , radius } \frac{1}{z_I}}$$

Summary

$$\text{Normalized load resistance : Circle with center } \left(\frac{z_R}{1+z_R}, 0\right) \text{ , radius } \frac{1}{1+z_R}$$

$$\text{Normalized load reactance : Circle with center } \left(1, \frac{1}{z_I}\right) \text{ , radius } \frac{1}{z_I}$$

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