The 3 Common Coordinate Systems

December 27, 2012

1 Summary

1.1 Rectangular

- Unit Vector $\hat{x}$, $\hat{y}$, $\hat{z}$
- Vector $V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$
- Magnitude $\sqrt{V_x^2 + V_y^2 + V_z^2}$
- Distance $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

1.2 Cylindrical

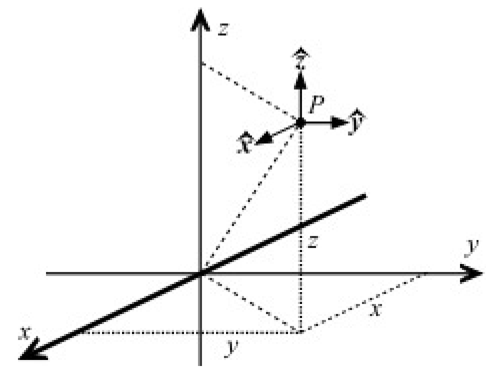
- Unit Vector $\hat{r}$, $\hat{\phi}$, $\hat{z}$
- Vector $V_r\hat{r} + V_\phi\hat{\phi} + V_z\hat{z}$
- Magnitude $\sqrt{V_r^2 + V_\phi^2 + V_z^2}$
- Distance $\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\phi_1 - \phi_2) + (z_1 - z_2)^2}$

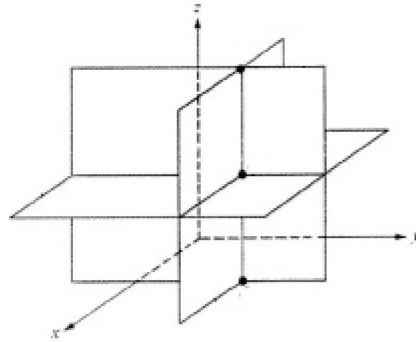
1.3 Spherical

- Unit Vector $\hat{R}$, $\hat{\phi}$, $\hat{\theta}$
- Vector $V_R\hat{R} + V_\phi\hat{\phi} + V_\theta\hat{\theta}$
- Magnitude $\sqrt{V_R^2 + V_\phi^2 + V_\theta^2}$
- Distance $\sqrt{R_1^2 + R_2^2 - 2R_1R_2\sin\theta_1\sin\theta_2\cos(\phi_1 - \phi_2) - 2R_1R_2\cos\theta_1\cos\theta_2}$

2 Rectangular Coordinate

Everything is simple in rectangular coordinate





$$-\infty \leq x \leq +\infty$$
$$-\infty \leq y \leq +\infty$$
$$-\infty \leq z \leq +\infty$$

Unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$

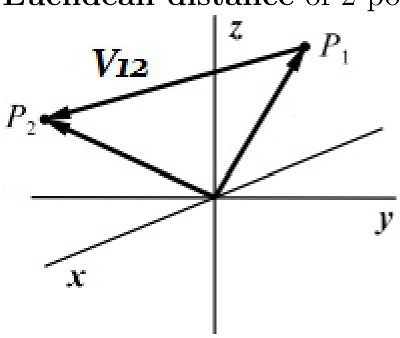
Vector Representation

$$\bar{V} = \bar{V}_x + \bar{V}_y + \bar{V}_z = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$$

Vector Magnitude

$$|\bar{V}| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

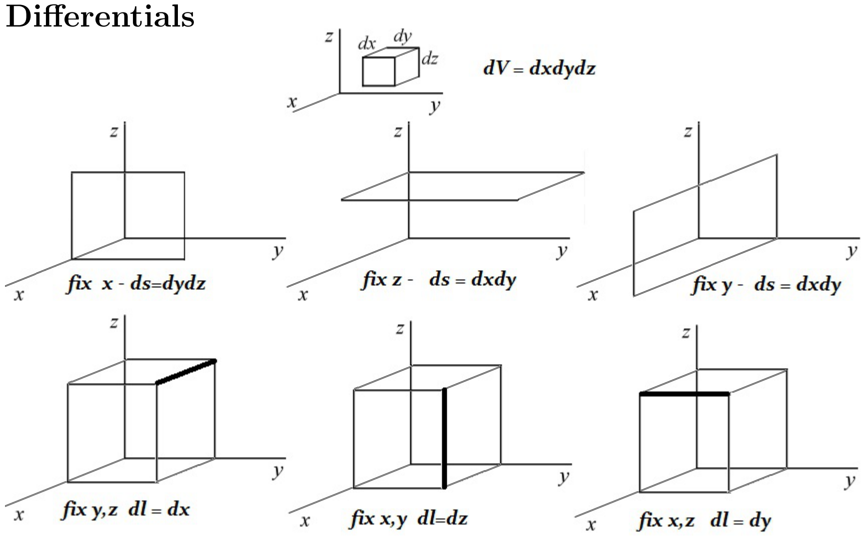
Euclidean distance of 2 points (x_1, y_1, z_1) , (x_2, y_2, z_2)



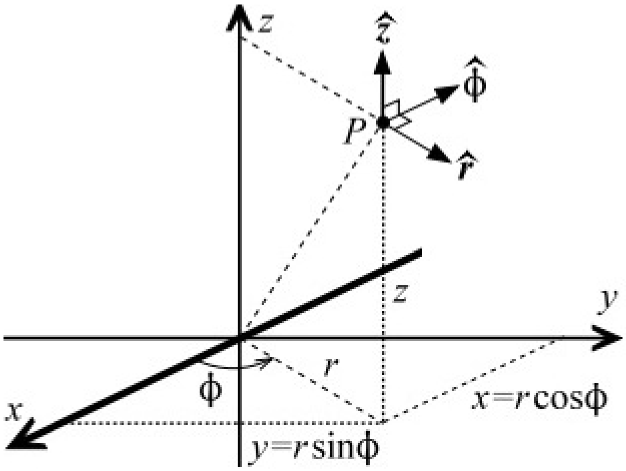
Just Pythagoras' theorem

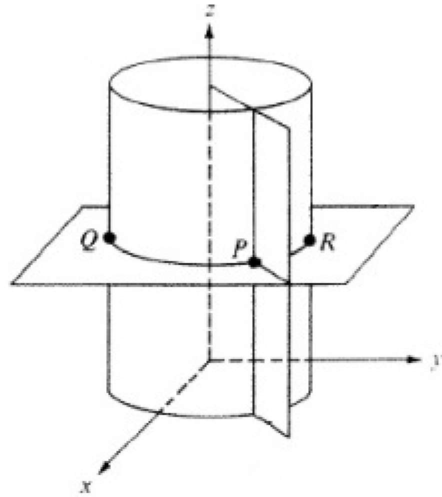
$$d = |\bar{V}_{12}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Differentials



3 Cylindrical Coordinate





$0 \leq r \leq \infty$
$0 \leq \phi \leq 2\pi$
$-\infty \leq z \leq +\infty$

Unit vector $\hat{r}$ **,** $\hat{\phi}$ **,** $\hat{z}$

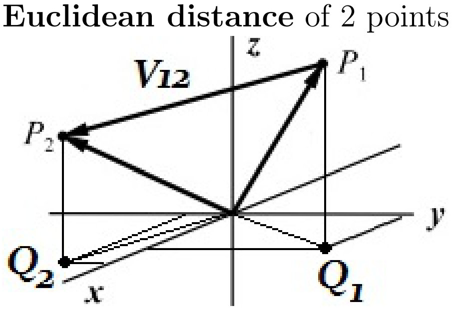
Vector representation

$$\bar{V} = \bar{V}_r + \bar{V}_\phi + \bar{V}_z = V_r\hat{r} + V_\phi\hat{\phi} + V_z\hat{z}$$

Vector Magnitude

$$|\bar{V}| = \sqrt{V_r^2 + V_\phi^2 + V_z^2}$$

Euclidean distance of 2 points (r_1, ϕ_1, z_1) , (r_2, ϕ_2, z_2)



The euclidean distance of cylindrical coordinate can be found by using

$$d_{12} = \sqrt{(\text{distance of projection Q})^2 + (\text{distance of height})^2} = \sqrt{d_{Q12}^2 + (z_1 - z_2)^2}$$

The distance d_{Q12} , which is in polar coordinate , can be found by using relation of polar coodinate and rectangular coordinate

$$d_{Q12} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Express as polar coordinate

$$= \sqrt{(r_1\cos\phi_1 - r_2\cos\phi_2)^2 + (r_1\sin\phi_1 - r_2\sin\phi_2)^2}$$

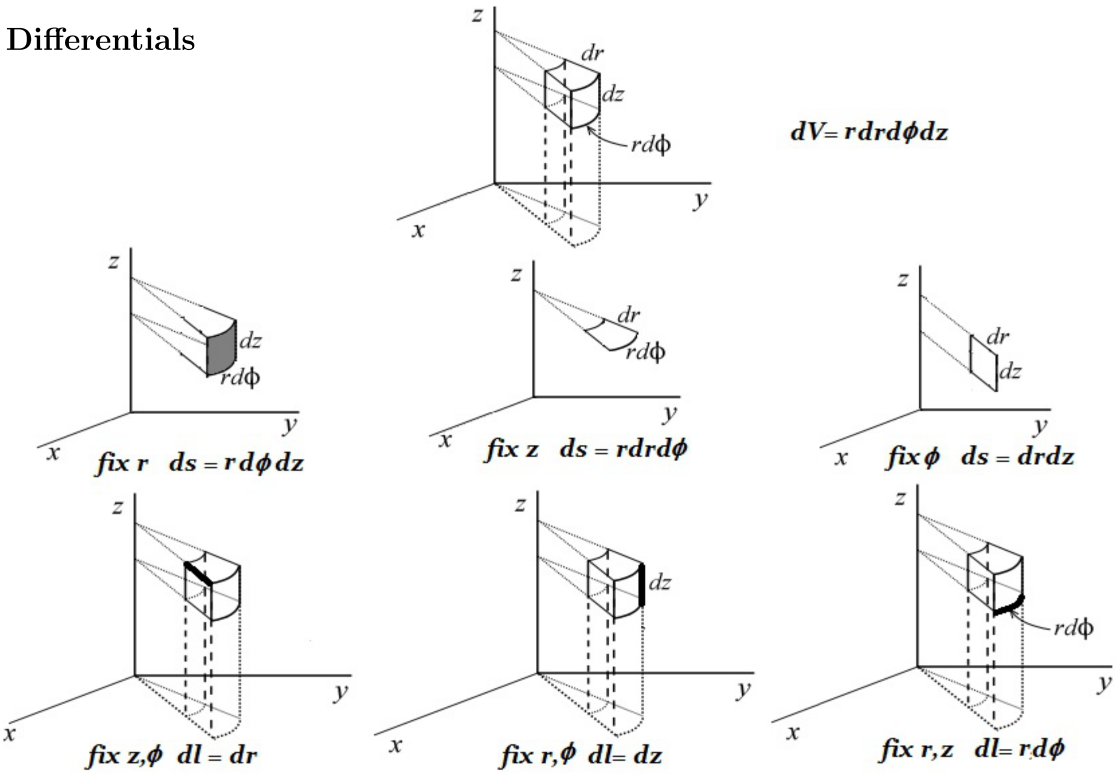
Extract the square term out to form $r^2\cos^2\phi + r^2\sin^2\phi = r^2$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1\cos\phi_1 r_2\cos\phi_2 - 2r_1\sin\phi_1 r_2\sin\phi_2} = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\phi_1 - \phi_2)}$$

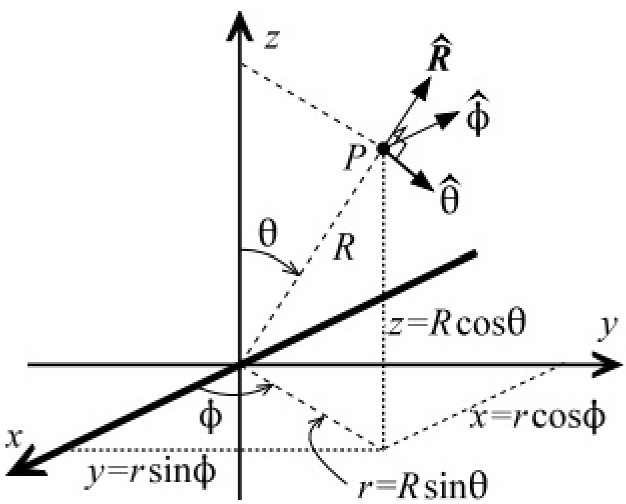
Thus, the euclidean distance of 2 points is

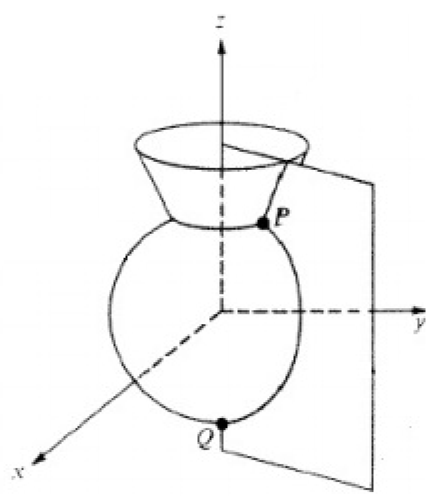
$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\phi_1 - \phi_2) + (z_1 - z_2)^2}$$

Differentials



4 Spherical Coordinate





$0 \leq R \leq +\infty$
$0 \leq \phi \leq 2\pi$
$0 \leq \theta \leq \pi$

Unit vector $\hat{R}$, $\hat{\phi}$, $\hat{\theta}$

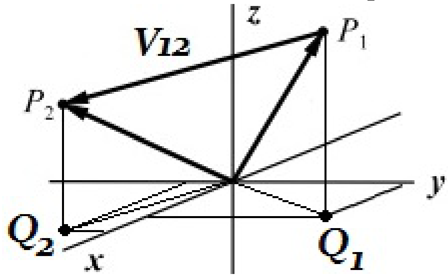
Vector representation

$$\bar{V} = \bar{V}_R + \bar{V}_\phi + \bar{V}_\theta = V_R\hat{R} + V_\phi\hat{\phi} + V_\theta\hat{\theta}$$

Vector Magnitude

$$|\bar{V}| = \sqrt{V_R^2 + V_\phi^2 + V_\theta^2}$$

Euclidean distance of 2 points (r_1, ϕ_1, θ_1) , (r_2, ϕ_2, θ_2)



Using same techniques, consider the projection distance and the height distance

$$d_{12} = \sqrt{\text{projection distance}^2 + \text{height distance}^2}$$

Where

$$\text{Projection distance } = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \qquad \text{Height distance} = \sqrt{(z_1 - z_2)^2}$$

Turn them into spherical equivalent

$$\text{Projection distance } = \sqrt{(R_1 \sin\theta_1 \cos\phi_1 - R_2 \sin\theta_2 \cos\phi_2)^2 + (R_1 \sin\theta_1 \sin\phi_1 - R_2 \sin\theta_2 \sin\phi_2)^2}$$

$$\text{Height distance } = \sqrt{(R_1 \cos\theta_1 - R_2 \cos\theta_2)^2}$$

$$d_{12} = \left[(R_1 \sin\theta_1 \cos\phi_1 - R_2 \sin\theta_2 \cos\phi_2)^2 + (R_1 \sin\theta_1 \sin\phi_1 - R_2 \sin\theta_2 \sin\phi_2)^2 + (R_1 \cos\theta_1 - R_2 \cos\theta_2)^2\right]^{\frac{1}{2}}$$

For the frist 2 term, extract the terms $R^2 \sin^2\theta \sin^2\phi + R^2 \sin^2\theta \cos^2\phi = R^2 \sin^2\theta$

Then combine these term with the term $R^2 \cos^2\theta$ from third bracket to form R^2

i.e.

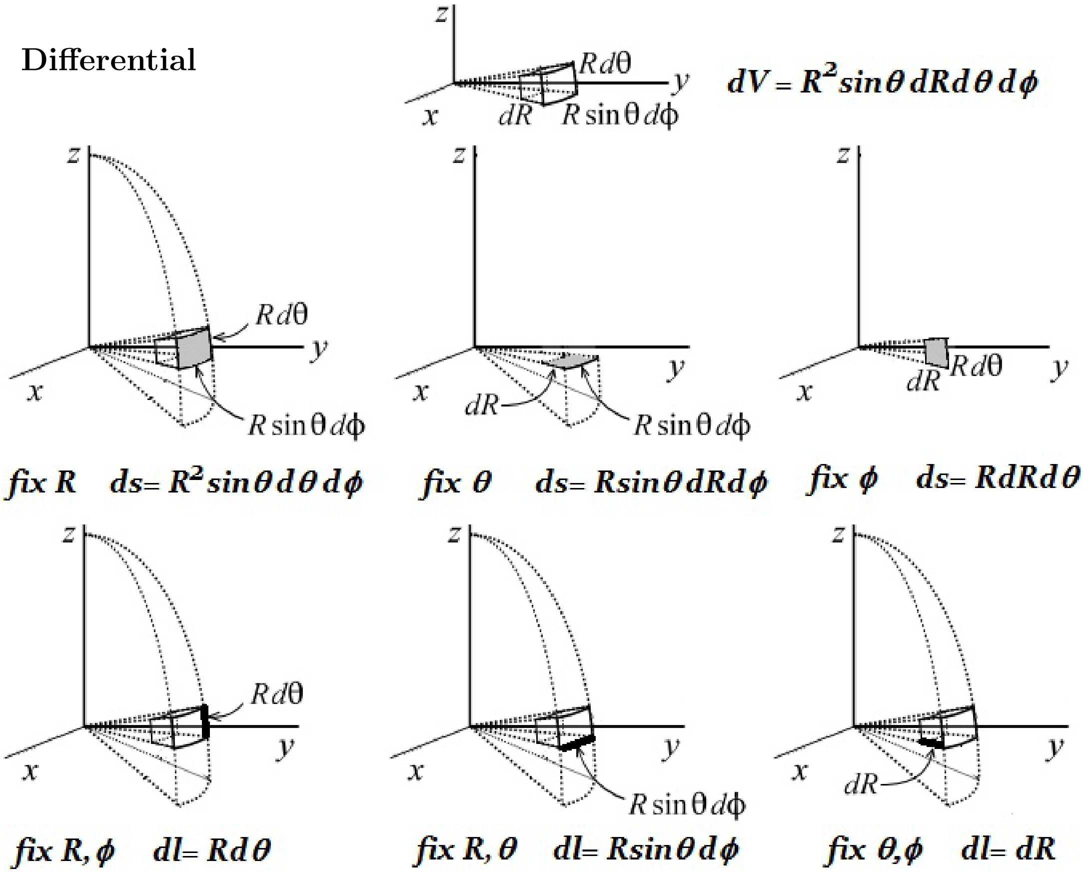
$$d_{12} = \left[(a-b)^2 + (c-d)^2 + (e-f)^2\right]^{\frac{1}{2}} = [(\text{Square terms}) - 2ab - 2cd - 2ef]^{\frac{1}{2}}$$

Where the square terms will combine to form R_1^2 and R_2^2

$$d_{12} = \sqrt{R_1^2 + R_2^2 - \underbrace{2R_1 \sin\theta_1 \cos\phi_1 R_2 \sin\theta_2 \cos\phi_2 - 2R_1 \sin\theta_1 \sin\phi_1 R_2 \sin\theta_2 \sin\phi_2} - 2R_1 R_2 \cos\theta_1 \cos\theta_2}$$

$$d_{12} = \sqrt{R_1^2 + R_2^2 - 2R_1 R_2 \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) - 2R_1 R_2 \cos\theta_1 \cos\theta_2}$$

Differential



5 Differential Transformation using Jacobian

Change the unit vector from $\hat{x}_1, \hat{x}_2, \hat{x}_3$ to $\hat{y}_1$, $\hat{y}_2$, $\hat{y}_3$, then the change of differential can be found by

$$dx_1 dx_2 dx_3 = \left| J\left(y_1, y_2, y_3\right) \right| dy_1 dy_2 dy_3 \qquad J\left(y_1, y_2, y_3\right) = \frac{\partial\left(x_1 x_2 x_3\right)}{\partial\left(y_1 y_2 y_3\right)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \\ \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix}$$

Rectangular to cylindrical , from x, y, z to r, ϕ, z ($x = r\cos\phi$, $y = r\sin\phi$, $z = z$)

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$dxdydz = rdrd\phi dz$$

Rectangular to spherical , from x, y, z to R, ϕ, θ ($x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$, $z = R\cos\theta$)

$$J=\begin{vmatrix}\frac{\partial x}{\partial R} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta}\\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\sin\theta\cos\phi & -R\sin\theta\sin\phi & R\cos\theta\cos\phi\\ \sin\theta\sin\phi & R\sin\theta\cos\phi & R\cos\theta\sin\phi\\ \cos\theta & 0 & -R\sin\theta\end{vmatrix}=R^2\begin{vmatrix}\sin\theta\cos\phi & -\sin\theta\sin\phi & \cos\theta\cos\phi\\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta\sin\phi\\ \cos\theta & 0 & -\sin\theta\end{vmatrix}$$

Expand along the second column

$$=R^2\left(\sin\theta\sin\phi\begin{vmatrix}\sin\theta\sin\phi & \cos\theta\sin\phi\\ \cos\theta & -\sin\theta\end{vmatrix}+\sin\theta\cos\phi\begin{vmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi\\ \cos\theta & -\sin\theta\end{vmatrix}\right)$$

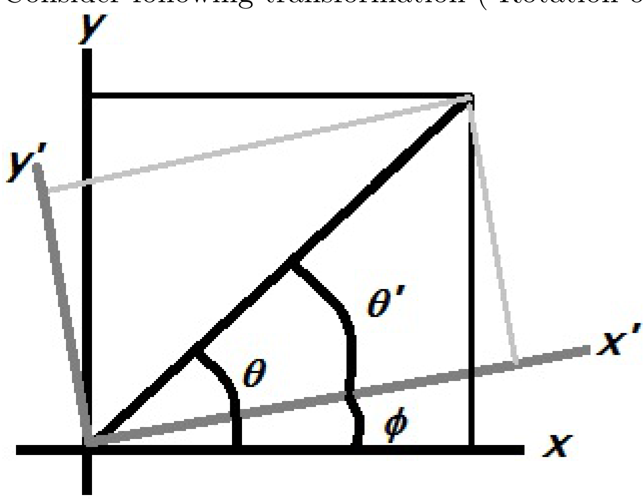
$$=R^2\left(-\sin\theta\sin^2\phi-\sin\theta\cos^2\phi\right)=-R^2\sin\theta$$

$$dxdydz=R^2\sin\theta dRd\theta d\phi$$

6 Vector Transformation using Geometry

General idea

Consider following transformation (Rotation of axis with angle ϕ)



$$V=V_x\hat{x}+V_y\hat{y}=V_{x'}\hat{x}'+V_{y'}\hat{y}' \qquad \begin{matrix}V_x=V\cos\theta\\ V_y=V\sin\theta\end{matrix} \qquad \begin{matrix}V_{x'}=V\cos\theta'=V\cos(\theta-\phi)\\ V_{y'}=V\sin\theta'=V\sin(\theta-\phi)\end{matrix}$$

$$\begin{matrix}V_{x'}=V\cos(\theta-\phi)=V\cos\theta\cos\phi+V\sin\theta\sin\phi=V_x\cos\phi+V_y\sin\phi\\ V_{y'}=V\sin(\theta-\phi)=V\sin\theta\cos\phi-V\cos\theta\sin\phi=V_y\cos\phi-V_x\sin\phi\end{matrix}$$

In Matrix notation

$$\begin{bmatrix}V_{x'}\\ V_{y'}\end{bmatrix}=\begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}V_x\\ V_y\end{bmatrix}$$

Rectangular to cylindrical , given the vector $\bar{V}=V_x\hat{x}+V_y\hat{y}+V_z\hat{z}$, express it as $\bar{V}=V_r\hat{r}+V_\phi\hat{\phi}+V_z\hat{z}$
To find the component vector, the key is to use dot product

$$V_r=\bar{V}\cdot\hat{r}=V_x\hat{x}\cdot\hat{r}+V_y\hat{y}\cdot\hat{r}+V_z\hat{z}\cdot\hat{r}=V_x\hat{x}\cdot\hat{r}+V_y\hat{y}\cdot\hat{r}$$

$$V_\phi=\bar{V}\cdot\hat{\phi}=V_x\hat{x}\cdot\hat{\phi}+V_y\hat{y}\cdot\hat{\phi}+V_z\hat{z}\cdot\hat{\phi}=V_x\hat{x}\cdot\hat{\phi}+V_y\hat{y}\cdot\hat{\phi}$$

$$V_z = \bar{V} \cdot \hat{z} = V_x \hat{x} \cdot \hat{\phi} + V_y \hat{y} \cdot \hat{\phi} + V_z \hat{z} \cdot \hat{z} = V_z$$

To find the dot product, find the $\hat{r}$ and $\hat{\phi}$ using geometry
The matrix equation can be used, since it is a 2D transformation (z-component equal)

$$\begin{bmatrix} \hat{r} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \qquad \Rightarrow \begin{matrix} \hat{r} = \cos\phi\hat{x} + \sin\phi\hat{y} \\ \hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \end{matrix}$$

Thus

$$V_r = V_x \hat{x} \cdot \hat{r} + V_y \hat{y} \cdot \hat{r} = V_x \cos\phi + V_y \sin\phi$$

$$V_\phi = V_x \hat{x} \cdot \hat{\phi} + V_y \hat{y} \cdot \hat{\phi} = -V_x \sin\phi + V_y \cos\phi$$

And therefore the vector $\bar{V} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$ has transformed into

$$\bar{V} = (V_x \cos\phi + V_y \sin\phi)\,\hat{r} + (-V_x \sin\phi + V_y \cos\phi)\,\hat{\phi} + V_z\hat{z}$$

In matrix form

$$\begin{bmatrix} V_r \\ V_\phi \\ V_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

Cylindrical to rectangular
The transformation can be done using inverse matrix

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} V_r \\ V_\phi \\ V_z \end{bmatrix}$$

$$\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}^{-1} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \qquad \text{So} \qquad \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_r \\ V_\phi \\ V_z \end{bmatrix}$$

Where

$$r = \sqrt{x^2 + y^2} \qquad \cos\phi = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}} \qquad \sin\phi = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

Therefore

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{-y}{\sqrt{x^2+y^2}} & 0 \\ \dfrac{y}{\sqrt{x^2+y^2}} & \dfrac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_r \\ V_\phi \\ V_z \end{bmatrix}$$

Thus for given $\bar{V} = V_r\hat{r} + V_\phi\hat{\phi} + V_z\hat{z}$

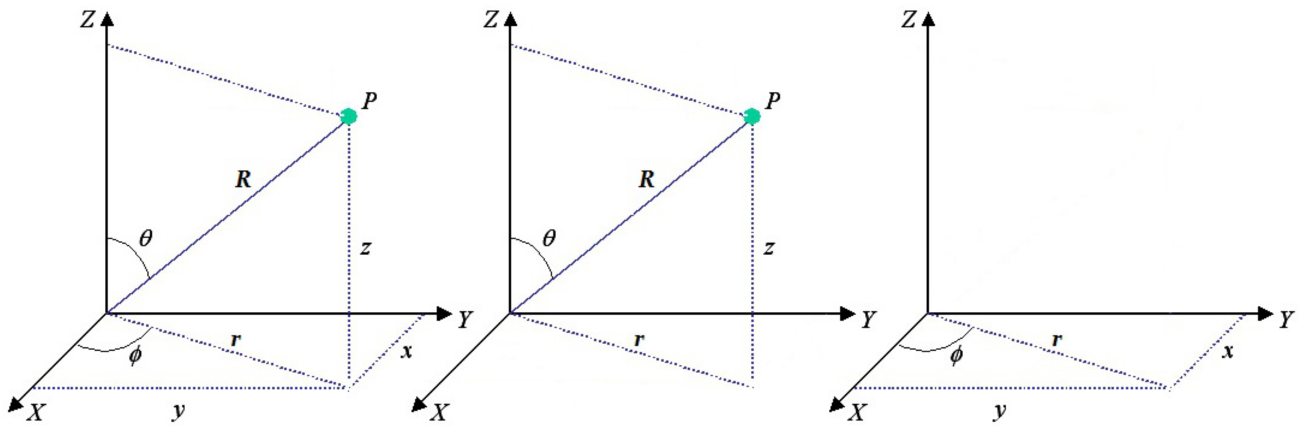
$$\bar{V} = \left(V_r \frac{x}{\sqrt{x^2+y^2}} - V_\phi \frac{y}{\sqrt{x^2+y^2}} \right) \hat{x} + \left(V_r \frac{y}{\sqrt{x^2+y^2}} + V_\phi \frac{x}{\sqrt{x^2+y^2}} \right) \hat{y} + V_z \hat{z}$$

Rectangular to spherical given the vector $\bar{V} = V_x\hat{x}+V_y\hat{y}+V_z\hat{z}$, express it as $\bar{V} = V_R\hat{R}+V_\phi\hat{\phi}+V_\theta\hat{\theta}$

$$V_R = \bar{V} \cdot \hat{R} = V_x\hat{x} \cdot \hat{R} + V_y\hat{y} \cdot \hat{R} + V_z\hat{z} \cdot \hat{R}$$

$$V_\phi = \bar{V} \cdot \hat{\phi} = V_x\hat{x} \cdot \hat{\phi} + V_y\hat{y} \cdot \hat{\phi} + V_z\hat{z} \cdot \hat{\phi}$$

$$V_\theta = \bar{V} \cdot \hat{\theta} = V_x\hat{x} \cdot \hat{\theta} + V_y\hat{y} \cdot \hat{\theta} + V_z\hat{z} \cdot \hat{\theta}$$



Fist, consider the projection plane (3rd figure)

$$\hat{r} = \cos\phi\hat{x} + \sin\phi\hat{y}$$

And then consider the 2nd figure

$$\hat{R} = \cos\theta\hat{z} + \sin\theta\hat{r}$$

$$\hat{R} = \cos\theta\hat{z} + \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y}$$

$$\hat{\theta} = \cos\theta\hat{r} + (-\sin(90-\theta))\hat{z}$$

$$= \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$$

$$\hat{\phi} = \cos\phi\hat{y} + (-\sin\phi)\hat{x}$$

i.e.

$$\begin{cases} \hat{R} = \cos\theta\hat{z} + \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} \\ \hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \\ \hat{\phi} = \cos\phi\hat{y} - \sin\phi\hat{x} \end{cases} \quad \text{and} \quad \begin{cases} V_R = \bar{V} \cdot \hat{R} = V_x\hat{x} \cdot \hat{R} + V_y\hat{y} \cdot \hat{R} + V_z\hat{z} \cdot \hat{R} \\ V_\phi = V_x\hat{x} \cdot \hat{\phi} + V_y\hat{y} \cdot \hat{\phi} + V_z\hat{z} \cdot \hat{\phi} \\ V_\theta = V_x\hat{x} \cdot \hat{\theta} + V_y\hat{y} \cdot \hat{\theta} + V_z\hat{z} \cdot \hat{\theta} \end{cases}$$

And

$$\begin{array}{ccc} \hat{x}\cdot\hat{R}=\sin\theta\cos\phi & \hat{x}\cdot\hat{\phi}=-\sin\phi & \hat{x}\cdot\hat{\theta}=\cos\theta\cos\phi \\ \hat{y}\cdot\hat{R}=\sin\theta\sin\phi & \hat{y}\cdot\hat{\phi}=\cos\phi & \hat{y}\cdot\hat{\theta}=\cos\theta\sin\phi \\ \hat{z}\cdot\hat{R}=\cos\theta & \hat{z}\cdot\hat{\phi}=0 & \hat{z}\cdot\hat{\theta}=-\sin\theta \end{array}$$

Thus

$$\begin{cases} V_R = V_x\sin\theta\cos\phi + V_y\sin\theta\sin\phi + V_z\cos\theta \\ V_\phi = -V_x\sin\phi + V_y\cos\phi \\ V_\theta = V_x\cos\theta\cos\phi + V_y\cos\theta\sin\phi - V_z\sin\theta \end{cases}$$

i.e. given vector $\bar{V} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$,

$$\hat{V} = (V_x\sin\theta\cos\phi + V_y\sin\theta\sin\phi + V_z\cos\theta)\,\hat{R} + (-V_x\sin\phi + V_y\cos\phi)\,\hat{\phi} + (V_x\cos\theta\cos\phi + V_y\cos\theta\sin\phi - V_z\sin\theta)\,\hat{\theta}$$

Or in matrix form

$$\begin{bmatrix} V_R \\ V_\phi \\ V_\theta \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

Spherical to rectangular

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{bmatrix}^{-1} \begin{bmatrix} V_R \\ V_\phi \\ V_\theta \end{bmatrix}$$

$$\begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{bmatrix}^{-1} = \begin{bmatrix} \sin\theta\cos\phi & \sin\phi & \cos\theta\cos\phi \\ \sin\theta\sin\phi & \cos\phi & \cos\theta\sin\phi \\ \cos\theta & 0 & -\sin\theta \end{bmatrix}$$

i.e.

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & -\sin\phi & \cos\theta\cos\phi \\ \sin\theta\sin\phi & \cos\phi & \cos\theta\sin\phi \\ \cos\theta & 0 & -\sin\theta \end{bmatrix} \begin{bmatrix} V_R \\ V_\phi \\ V_\theta \end{bmatrix}$$

i.e.

$$\begin{array}{c} V_x = \sin\theta\cos\phi V_R - \sin\phi V_\phi + \cos\theta\cos\phi V_\theta \\ V_y = \sin\theta\sin\phi V_R + \cos\phi V_\phi + \cos\theta\sin\phi V_\theta \\ V_z = \cos\theta V_R - \sin\theta V_\theta \end{array}$$

Where

$$\sin\theta = \frac{r}{R} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \quad \cos\theta = \frac{z}{R} = \frac{z}{\sqrt{x^2+y^2+z^2}} \quad \sin\phi = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}} \quad \cos\phi = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

i.e.

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{-y}{\sqrt{x^2+y^2}} & \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{x}{\sqrt{x^2+y^2}} & \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & 0 & \frac{-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \end{bmatrix} \begin{bmatrix} V_R \\ V_\phi \\ V_\theta \end{bmatrix}$$

i.e. given vector $\bar{V} = V_R\hat{R} + V_\phi\hat{\phi} + V_\theta\hat{\theta}$

$$\begin{aligned}\bar{V} = \quad & \left(\frac{x}{\sqrt{x^2+y^2+z^2}}V_R + \frac{-y}{\sqrt{x^2+y^2}}V_\phi + \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}V_\theta\right)\hat{x} \\ & + \left(\frac{y}{\sqrt{x^2+y^2+z^2}}V_R + \frac{x}{\sqrt{x^2+y^2}}V_\phi + \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}V_\theta\right)\hat{y} \\ & + \left(\frac{z}{\sqrt{x^2+y^2+z^2}}V_R + \frac{-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}V_\theta\right)\hat{z}\end{aligned}$$