Derivations of Optics Laws

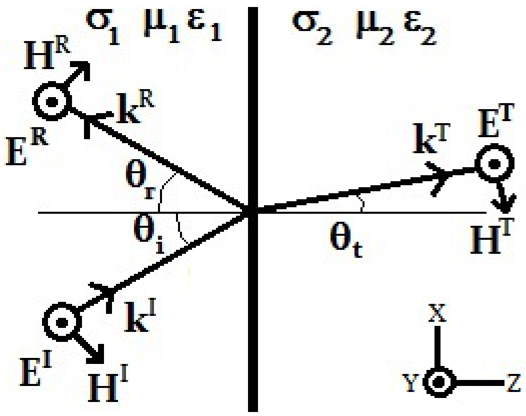
April 26, 2013

1 The laws

Law of Reflection $\theta_1 = \theta_2$

Law of Refraction (Snell's Law) $\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{n_2}{n_1} = \dfrac{v_1}{v_2} = \dfrac{\lambda_1}{\lambda_2}$

2 Derivation from Electromagnetic Wave Propagation in 2 medium



$$k_1 = \omega\sqrt{\mu_1\varepsilon_1}$$
$$k_2 = \omega\sqrt{\mu_2\varepsilon_2}$$

$$k^I = k_{ix}\hat{x} + k_{iz}\hat{z}$$
$$k^R = k_{rx}\hat{x} + k_{rz}\hat{z}$$
$$k^T = k_{tx}\hat{x} + k_{tz}\hat{z}$$

$$|k^I| = |k_1| \ , \ |k^R| = |k_1| \ , \ |k^T| = |k_2|$$

$$k_{ix} = k_1\sin\theta_i \quad k_{rx} = k_1\sin\theta \quad k_{tx} = k_2\sin\theta$$
$$k_{iz} = k_1\cos\theta_i \quad k_{rz} = -k_1\cos\theta \quad k_{tz} = k_2\cos\theta$$

$$E^I = E_0\exp\left(k_1\sin\theta_i x + k_1\cos\theta_i z\right)\hat{y}$$
$$E^R = RE_0\exp\left(k_1\sin\theta_r x - k_1\cos\theta_r z\right)\hat{y}$$
$$E^T = TE_0\exp\left(k_2\sin\theta_t x + k_2\cos\theta_t z\right)\hat{y}$$

Apply the boundary condition at $z = 0$

$E^I_{z=0} + E^R_{z=0} = E^T_{z=0}$

$E_0\exp\left(k_1\sin\theta_i x + k_1\cos\theta_i z\right) + RE_0\exp\left(k_1\sin\theta_r x - k_1\cos\theta_r z\right) = TE_0\exp\left(k_2\sin\theta_t x + k_2\cos\theta_t z\right)$

$\exp\left(k_1\sin\theta_i x\right) + R\exp\left(k_1\sin\theta_r x\right) = T\exp\left(k_2\sin\theta_t x\right)$

In the case $R = -1$, $T = 0$

$\exp\left(k_1\sin\theta_i x\right) = \exp\left(k_1\sin\theta_r x\right)$

Thus $\sin\theta_i = \sin\theta_r$

$$\theta_i = \theta_r$$

In the case $R = 0$, $T = 1$

$\exp\left(k_1\sin\theta_i x\right) = \exp\left(k_2\sin\theta_t x\right)$

Thus $k_1\sin\theta_i = k_2\sin\theta_t$

$\omega\sqrt{\mu_1\varepsilon_1}\sin\theta_i = \omega\sqrt{\mu_2\varepsilon_2}\sin\theta_t$

Since $n = \sqrt{\mu\varepsilon}$

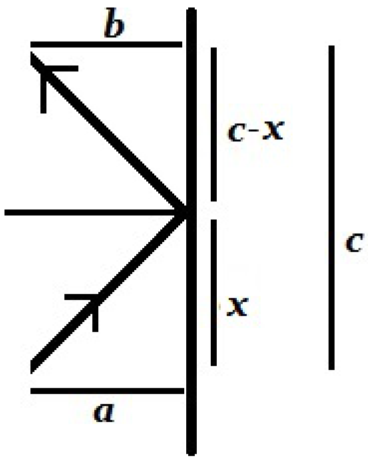
$$n_1\sin\theta_1 = n_2\sin\theta_t$$

3 Derivation from Fermat's Principle , an optimization approach

Fermat's Principle

Light follows the path of the least time

Reflection



Total pathlength of light travelled

$$L(x) = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c - x)^2}$$

The time taken for the wave travelled

$$t = \frac{L}{v} \propto L$$

Fermat Principle $\Rightarrow \min t = \min L$

To obtain minimizer $x*$

$$\frac{\partial L}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{a^2 + x^2}} + \frac{1}{2}\frac{2(c - x)(-1)}{\sqrt{b^2 + (c - x)^2}} = 0$$

$$\frac{x}{\sqrt{a^2 + x^2}} = \frac{x - c}{\sqrt{b^2 + (c - x)^2}}$$

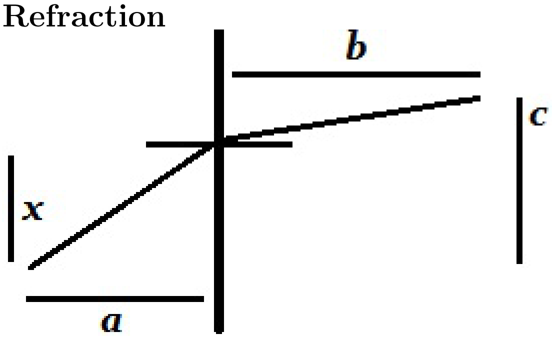
This is exactly

$$\sin\theta_i = \sin\theta_r$$

i.e.

$$\theta_i = \theta_r$$

Refraction



$$L(x) = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c - x)^2}$$

$$t = \frac{L}{v}$$

$$\frac{\partial t}{\partial x} = \frac{1}{2v_1}\frac{2x}{\sqrt{a^2 + x^2}} + \frac{1}{2v_2}\frac{2(c - x)(-1)}{\sqrt{b^2 + (c - x)^2}} = 0$$

$$\frac{x}{v_1\sqrt{a^2 + x^2}} = \frac{(c - x)}{v_2 2\sqrt{b^2 + (c - x)^2}}$$

This is

$$\frac{\sin\theta_i}{v_1} = \frac{\sin\theta_t}{v_2}$$

i.e.

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{v_1}{v_2}$$

$-END-$