Material Dispersion

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Recall that reflective index is the ratio of the speed of EM wave in free space and medium

$$n = \frac{c}{v} = \frac{\frac{1}{\sqrt{\varepsilon_0 \mu_0}}}{\frac{1}{\sqrt{\varepsilon \mu}}} = \frac{\sqrt{\varepsilon_r \varepsilon_0 \mu_r \mu_0}}{\sqrt{\varepsilon_0 \mu_0}} = \sqrt{\varepsilon_r \mu_r}$$

Where ε_r and μ_r is the relative permittivity and relative permeability (relative permittivity is also called dielectric constant)

For *dispersive material* , $\varepsilon_r \mu_r$ are functions of wavelength , that is, the ε_r , μ_r will change as λ change !

$$n = \sqrt{\varepsilon_r(\lambda)\mu_r(\lambda)}$$

In Engineering Electromagnetics, the dispersive relation is often expressed as circular frequency

$$\varepsilon_r(\omega) \qquad \mu_r(\omega)$$

While in Fiber Optics, the dispersive relation is expressed as wavelength.

Since the reflective index is different for different wavelength, thus waves with different wavelength travel with different speed

For example, consider a wave with 2 component $W = W(\lambda_1) + W(\lambda_2)$, since $\lambda_1 \neq \lambda_2$, thus the 2 wave will travel with different speed.

When $\lambda_1 \approx \lambda_2$ (the 2 components are in similar wavelength or similar frequency) , then $v_1 \approx v_2$, in this case, **since their speed is approximately the same, and thus superposition of 2 waves with very close frequencies will form a "beat"**

The beat phenomenon, phase velocity and group velocity

Recall from high school physics, when 2 waves with similar frequencies $\omega \pm \Delta\omega$ and with similar wavevector $k \pm \Delta k$, and same amplitude interference with each other

$$E = E_1 + E_2 = E_0 \left(\cos\left[(\omega + \Delta\omega)\, t - (k + \Delta k)\, \Delta z \right] + \cos\left[(\omega - \Delta\omega)\, t - (k - \Delta k)\, \Delta z \right] \right)$$

$$E = 2E_0 \cos\left(\Delta\omega t - \Delta k z\right) \cos\left(\omega t - kz\right)$$

Since $\Delta\omega$ should be very small (otherwise 2 waves are not "similar" frequencies) , thus the first cosine term should be slow changing (envelope), and the second one is fast changing (oscillation)

Consider constant phase for the oscillation term

$$\omega t - kz = \text{const}$$

Thus the phase velocity v_p is thus

$$v_p = \frac{dz}{dt} = \frac{\omega}{k}$$

Consider constant phase for the envelope term

$$\Delta\omega t - \Delta k z = \text{const}$$

Thus the group velocity v_g is thus

$$v_g = \frac{\Delta\omega}{\Delta k}$$

Consider $\Delta\omega$, Δk very samll

$$v_g = \frac{d\omega}{dk}$$

In dispersive medium

$$v_g = \frac{d\omega}{dk} = \frac{1}{\frac{dk}{d\omega}}$$

Consider phase velocity relation

$$\frac{c}{n(\lambda)} = v_p = \frac{\omega}{k} \qquad \rightarrow k = \frac{\omega n(\lambda)}{c}$$

And thus

$$\frac{dk}{d\omega} = \frac{n(\lambda)}{c} + \frac{\omega}{c}\frac{dn(\lambda)}{d\omega}$$

Thus

$$v_g = \frac{1}{\frac{dk}{d\omega}} = \frac{c}{n(\lambda) + \omega\dfrac{dn(\lambda)}{d\omega}}$$

We can express $\frac{dn(\lambda)}{d\omega}$ as $\frac{d(\lambda)}{d\lambda}$

$$\frac{dn(\lambda)}{d\omega} = \frac{dn(\lambda)}{d\lambda}\frac{d\lambda}{d\omega} = \frac{d\lambda}{d\omega}\frac{dn(\lambda)}{d\lambda}$$

Express λ as ω

$$\lambda = \frac{c}{f} = \frac{2\pi c}{2\pi f} = \frac{2\pi c}{\omega}$$

Thus

$$\frac{d\lambda}{d\omega} = -\frac{2\pi c}{\omega^2} = -\frac{1}{\omega}\frac{2\pi c}{\omega} = -\frac{\lambda}{\omega}$$

Thus

$$\frac{dn(\lambda)}{d\omega} = -\frac{\lambda}{\omega}\frac{dn(\lambda)}{d\lambda}$$

$$v_g = \frac{1}{\frac{dk}{d\omega}} = \frac{c}{n(\lambda) - \lambda\dfrac{dn(\lambda)}{d\lambda}}$$

Since the relationship between phase velocity and refractive index is $v_p = \frac{c}{n}$, thus $n(\lambda) - \lambda\frac{dn(\lambda)}{d\lambda}$ can be consider as "group reflective index" and thus v_g has same form as v_p

$$N(\lambda) = n(\lambda) - \lambda\frac{dn(\lambda)}{d\lambda}$$

Group Delay

The v_g characterize the velocity of the group of waves, and thus called group velocity.

Then the time that required for the group of wave to travel through 1 unit length is called *group delay*

$$\tau_g = \frac{1}{v_g} = \frac{n(\lambda) - \lambda\frac{dn(\lambda)}{d\lambda}}{c} = \frac{N(\lambda)}{c}$$

Then the time that required for the group of wave to travel through length L

$$\tau = \frac{L}{c} = \frac{LN(\lambda)}{c}$$

For a light pulse with spectral width $\Delta\lambda$

$$\Delta\tau = \tau_1 - \tau_2 = \frac{L}{c}\Big(N(\lambda_1) - N(\lambda_2)\Big) = \frac{L}{c}\Delta N(\lambda)$$

Divide both side with $\Delta\lambda$

$$\frac{\Delta\tau}{\Delta\lambda} = \frac{L}{c}\frac{\Delta N(\lambda)}{\Delta\lambda} = \frac{L}{c}\frac{dN(\lambda)}{d\lambda}$$

Expand the $\frac{dN}{d\lambda}$

$$\frac{dN(\lambda)}{d\lambda} = \frac{d}{d\lambda}\left(n(\lambda) - \lambda\frac{dn}{d\lambda}\right) = \frac{dn(\lambda)}{d\lambda} - \frac{d}{d\lambda}\left(\lambda\frac{dn(\lambda)}{d\lambda}\right) = \frac{dn(\lambda)}{d\lambda} - \frac{dn(\lambda)}{d\lambda} - \frac{dn(\lambda)}{d\lambda} - \lambda\frac{d^2n(\lambda)}{d\lambda^2} = -\lambda\frac{d^2n(\lambda)}{d\lambda^2}$$

Thus

$$\frac{\Delta\tau}{\Delta\lambda} = \frac{L}{c}\frac{\Delta N(\lambda)}{\Delta\lambda} = -\frac{L}{c}\lambda\frac{d^2n(\lambda)}{d\lambda^2}$$

Thus the pulse spread $\Delta\tau$

$$\Delta\tau = -\frac{L\lambda}{c}\frac{d^2n(\lambda)}{d\lambda^2}\Delta\lambda$$

Define

$$D = -\frac{\lambda}{c}\frac{d^2n(\lambda)}{d\lambda^2}$$

As material dispersion , notice that the unit of material dispersion is ps/km-nm

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