Some points about Electrostatics

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1. **Deriving Colomb's Law of electrostatic field from Maxwell's Equation**

   Using Integral Form of Gauss's Law

$$\iint \bar{E} \cdot d\bar{S} = \frac{Q_{encl.}}{\epsilon}$$

   For spherical surface with radius r, it is symmetric, so integrand $\bar{E}$ can be taken out

$$\iint \bar{E} \cdot d\bar{S} = \bar{E} \cdot \iint d\bar{S} = \bar{E} \cdot 4\pi r^2 \hat{r}$$

   Thus

$$\bar{E}(r) = \frac{Q}{4\pi\epsilon r^2}\hat{r}$$

   Using relation $\bar{F} = q\bar{E}$, the Coulomb's Force Law

$$\bar{F}_{12} = \frac{1}{4\pi\epsilon}\frac{Q_1Q_2}{r_{12}^2}\hat{r}_{12} = \frac{Q_1Q_2}{4\pi\epsilon}\frac{\bar{r}_1 - \bar{r}_2}{|\bar{r}_1 - \bar{r}_2|^3}$$

2. **Derivation of Electrostatic Field is conservative**

   Consider work done of E-field on a charge in a close loop

$$work\,done = -Q\oint \bar{E} \cdot d\bar{l} = \frac{-QQ}{4\pi\epsilon}\oint \frac{\hat{r}}{r^2} \cdot \hat{n}dl = \frac{-QQ}{4\pi\epsilon}\oint \frac{dl}{r^2} = \frac{-QQ}{4\pi\epsilon}\int_P^P \left(-\frac{dr}{r}\right) = 0$$

   i.e.

   - Electrostatic Field is conservative field
   - Electrostatic force is conservative force

2. **Electrostatic Field is divergent field**

   Consider the previous result $\oint \bar{E} \cdot d\bar{l} = 0$, apply Stoke's Theorem $\oint \bar{E} \cdot d\bar{l} = \iint_S \nabla \times \bar{E} \cdot d\bar{S} = 0$, thus

$$\nabla \times \bar{E} = 0$$

   - This equation is true since there is no dynamic B-field. For electro-"static" problem, all term with $\frac{\partial}{\partial t} = 0$.
   - Note that the general Faraday's Law in Maxwell's Eqaution is $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$, with the dynamic time-varying B-field.

3. **Potential**

In smple case $V_B - V_A = \Delta V = -\int_A^B \bar{E} \cdot d\bar{l}$ or equivalently $\bar{E} = -\frac{dV}{dl}\hat{l}$. And generally $\bar{E} = -\nabla V$

4. **Potential - Poisson Integral**

For the discrete charge distribution

$$V = \frac{1}{4\pi\epsilon} \sum_{n=1}^{N} \frac{q_n}{r_n}$$

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For continuous charge distribution

$$V = \frac{1}{4\pi\epsilon} \int_C \frac{\rho_l(\bar{r}')}{|\bar{r} - \bar{r}'|} dl = \frac{1}{4\pi\epsilon} \iint_S \frac{\rho_s(\bar{r}')}{|\bar{r} - \bar{r}'|} dS = \frac{1}{4\pi\epsilon} \iiint_V \frac{\rho_l(\bar{r}')}{|\bar{r} - \bar{r}'|} dV$$

5. **Deriving Gauss's Law from Coulomb's Law**

Consider the Poisson Integral

$$\bar{E} = \frac{1}{4\pi\epsilon} \iiint_V \rho(\bar{r}') \frac{\bar{r} - \bar{r}'}{|\bar{r} - \bar{r}'|^3} dV$$

Consider a vector identity

$$\nabla \cdot \left(\frac{\hat{r}}{|\bar{r}|^3} \right) = 4\pi\delta(\bar{r})$$

Then

$$\nabla \cdot \bar{E} = \frac{1}{4\pi\epsilon} \iiint_V \rho(\bar{r}') 4\pi\delta(\bar{r}) dV = \frac{1}{\epsilon} \iiint_V \rho(\bar{r}')\delta(\bar{r}) dV = \frac{\rho(\bar{r}')}{\epsilon} \iiint_V \delta(\bar{r}) dV = \frac{\rho(\bar{r}')}{\epsilon}$$

6. **Gauss's Law**

$$\Phi_E = \bar{E} \cdot \overline{Total\ Area}_{\perp} = \oiint_S \bar{E} \cdot d\bar{S}$$

7. **Gauss's Law + spherical coordinate**

$$\Phi_E = \oiint_S \bar{E} \cdot d\bar{S} = \int_0^{2\pi} \int_0^{\pi} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \cdot \left(\hat{r}\, r^2 \sin\theta d\theta d\phi \right) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin d\theta \frac{q}{4\pi\epsilon_0} = \frac{q}{\epsilon}$$

8. **Coulomb's Law**

$$\bar{F}_{12} = \frac{1}{4\pi\epsilon_o} \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12} = \frac{1}{4\pi\epsilon_o} \frac{q_1 q_2}{|\bar{r}_1 - \bar{r}_2|^3} (\bar{r}_{01} - \bar{r}_{02})$$

9. **Explicit Coulomb's Law in rectangular coordinate**

$$\bar{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_o} \frac{\langle x_1 - x_2\, ,\, y_1 - y_2\, ,\, z_1 - z_2 \rangle}{\left| (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \right|^{\frac{3}{2}}}$$

10. Principle of Superposition for many discrete charges

$$\bar{F}_{all} = \sum \bar{F}_i = \frac{q_a}{4\pi\epsilon_0} \sum \frac{q_i}{r_{ai}^2} \hat{r}_{ai} = \frac{q_a}{4\pi\epsilon_0} \sum \frac{q_i}{|\bar{r}_a - \bar{r}_i|^3} (\bar{r}_a - \bar{r}_i)$$

11. Electric field for many discrete charges

$$\bar{E}_{all} = \frac{1}{4\pi\epsilon_0} \sum \frac{q_i}{r_{ai}^2} \hat{r}_{ai} = \frac{1}{4\pi\epsilon_0} \sum \frac{q_i}{|\bar{r}_a - \bar{r}_i|^3} (\bar{r}_a - \bar{r}_i)$$

12. Continuous charge distribution : linear, area, volume $dq = \rho_l dl = \rho_s dS = \rho_V dV$

13. Total charge and charge density

$$Q = \int dq = \int_C \rho_l dl = \iint_S \rho_s dS = \iiint_V \rho_V dV$$

14. E-field for continuous charge distribution

$$\bar{E}(\bar{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{(\bar{r} - \bar{r}')}{|\bar{r} - \bar{r}'|^3} dq$$

15. Explicit eqaution of E-field for volume charge distribution

$$\bar{E}(\bar{r}) = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\langle x_1 - x_2 \, , \, y_1 - y_2 \, , \, z_1 - z_2 \rangle}{\left| (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \right|^{\frac{3}{2}}} \rho \, (x', y', z') \, dx' dy' dz'$$