Laplace Equations in Electrostatics

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1 Derivation of Laplace Equation

The Gauss's Law

$$\nabla \cdot D = \rho$$

Plug in $\bar{D} = \varepsilon \bar{E}$

$$\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon}$$

Since $\bar{E} = -\nabla V$, thus

$$\nabla \cdot \nabla V = -\frac{\rho}{\varepsilon}$$

Consider $\nabla \cdot \nabla$

In rectangular coordinate (x, y, z)

$$\nabla \cdot \nabla = div \cdot grad = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \cdot \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In cylindrical coordinate (r, ϕ, z)

$$\nabla \cdot \nabla = \left(\frac{1}{r}\frac{\partial}{\partial r} r\hat{r} + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}\right) \cdot \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}\right)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

In spherical coordinate (r, θ, ϕ)

$$\nabla \cdot \nabla = \left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2\hat{r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\sin\theta\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}\right) \cdot \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Thus the **Poisson Equations** are

$$-\frac{\rho}{\varepsilon} = \nabla^2 V = \begin{cases} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V \\ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\right) V \\ \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right) V \end{cases}$$

i.e.

$$\begin{cases} \frac{\partial^2 V(x,y,z)}{\partial x^2} + \frac{\partial^2 V(x,y,z)}{\partial y^2} + \frac{\partial^2 V(x,y,z)}{\partial z^2} = -\frac{\rho}{\varepsilon} \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V(r,\phi,z)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V(r,\phi,z)}{\partial \phi^2} + \frac{\partial^2 V(r,\phi,z)}{\partial z^2} = -\frac{\rho}{\varepsilon} \\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V(r,\phi,\theta)}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V(r,\phi,\theta)}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V(r,\phi,\theta)}{\partial \phi^2} = -\frac{\rho}{\varepsilon} \end{cases}$$

Next, before solving these PDEs, let's review on general solutions of the ODEs.

2 General Solutions of ODEs

Type I

$$\frac{dy}{dx} + ay = 0 \qquad y = ke^{-\int a dx}$$

Type II

$$\frac{d^2y}{dx^2} = b^2 y \qquad y = Ae^{bx} + Be^{-bx}$$

Or using $\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$

$$y = A' \cosh bx + B' \sinh bx$$

They are equivalent

$$y = A'\frac{e^{bx} + e^{-bx}}{2} + B'\frac{e^{bx} - e^{-bx}}{2}$$

$$= \frac{A' + B'}{2}e^{bx} + \frac{A' - B'}{2}e^{-bx}$$

$$A = \frac{A' + B'}{2} \qquad B = \frac{A' - B'}{2}$$

Type III

$$\frac{d^2y}{dx^2} = -m^2 y \qquad y = Ae^{jbx} + Be^{-jbx}$$

Or using $\cos x = \dfrac{e^{jx} + e^{-jx}}{2}$, $\sin x = \dfrac{e^{jx} - e^{-jx}}{2j}$

$$y = A' \cos mx + B' \sin mx$$

They are equivalent

$$\begin{aligned} y &= A' \frac{e^{jmx} + e^{-jmx}}{2} + B' \frac{e^{jmx} - e^{-jmx}}{2j} \\ &= \frac{A' - jB'}{2} e^{jmx} + \frac{A' + jB'}{2} e^{-bx} \end{aligned}$$

Hence, to solve the following Laplace Equations ($\rho = 0$)

$$\begin{cases} \dfrac{\partial^2 V(x,y,z)}{\partial x^2} + \dfrac{\partial^2 V(x,y,z)}{\partial y^2} + \dfrac{\partial^2 V(x,y,z)}{\partial z^2} = 0 \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r \dfrac{\partial V(r,\phi,z)}{\partial r} \right) + \dfrac{1}{r^2} \dfrac{\partial^2 V(r,\phi,z)}{\partial \phi^2} + \dfrac{\partial^2 V(r,\phi,z)}{\partial z^2} = 0 \\ \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial V(r,\phi,\theta)}{\partial r} \right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta \dfrac{\partial V(r,\phi,\theta)}{\partial \theta} \right) + \dfrac{1}{r^2 \sin^2\theta} \dfrac{\partial^2 V(r,\phi,\theta)}{\partial \phi^2} = 0 \end{cases}$$

We need to following

$$\begin{cases} \dfrac{dy}{dx} = -ay \qquad y = ke^{-\int a dx} \\ \dfrac{d^2y}{dx^2} = b^2 y \qquad y = Ae^{bx} + Be^{-bx} = A' \cosh bx + B' \sinh bx \\ \dfrac{d^2y}{dx^2} = -m^2 y \qquad y = Ae^{jbx} + Be^{-jbx} = A' \cos mx + B' \sin mx \end{cases}$$

And a techniques "Separation of Variables"

3 Separation of Variables in Rectangular Coordinate

$$\frac{\partial^2 V(x,y,z)}{\partial x^2} + \frac{\partial^2 V(x,y,z)}{\partial y^2} + \frac{\partial^2 V(x,y,z)}{\partial z^2} = 0$$

First, let $V(x,y,z) = X(x)Y(y)Z(z)$
Thus

$$\frac{\partial^2 X(x)Y(y)Z(z)}{\partial x^2} + \frac{\partial^2 X(x)Y(y)Z(z)}{\partial y^2} + \frac{\partial^2 X(x)Y(y)Z(z)}{\partial z^2} = 0$$

Divide whole equation by $X(x)Y(y)Z(z)$

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(x)}\frac{\partial^2 Y(y)}{\partial y^2} + \frac{1}{Z(x)}\frac{\partial^2 Z(z)}{\partial z^2} = 0$$

Now the PDEs reduce to 3 ODEs

$$\underbrace{\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}}_{X-only}+\underbrace{\frac{1}{Y(x)}\frac{d^2Y(y)}{dy^2}}_{Y-only}+\underbrace{\frac{1}{Z(x)}\frac{d^2Z(z)}{dz^2}}_{Z-only}=0$$

Since 3 parts are one variable only, the other terms can be treated as constants, thus

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=a^2 \qquad \frac{1}{Y(x)}\frac{d^2Y(y)}{dy^2}=b^2 \qquad \frac{1}{Z(x)}\frac{d^2Z(z)}{dz^2}=-c^2 \qquad \text{where } c^2=a^2+b^2$$

Thus

$$\frac{d^2X(x)}{dx^2}=a^2X(x) \qquad \frac{d^2Y(y)}{dy^2}=b^2Y(y) \qquad \frac{d^2Z(z)}{dz^2}=-c^2Z(z)$$

Therefore , apply the ODEs solutions

$$X(x)=\begin{cases} Ae^{ax}+Be^{-ax} \\ A\cosh ax+B\sinh ax \end{cases} \qquad Y(y)=\begin{cases} Ce^{bx}+De^{-bx} \\ C\cosh bx+D\sinh bx \end{cases} \qquad Z(z)=\begin{cases} Ee^{jcz}+Fe^{-jcz} \\ E\cos cz+F\sin cz \end{cases}$$

And thus the general solution of the $V=XYZ$ is

$$V(x,y,z)=\begin{cases} \left(Ae^{ax}+Be^{-ax}\right)\left(Ce^{bx}+De^{-bx}\right)\left(Ee^{jcz}+Fe^{-jcz}\right) \\ \left(A\cosh ax+B\sinh ax\right)\left(C\cosh bx+D\sinh bx\right)\left(E\cos cz+F\sin cz\right) \end{cases}$$

And the A,B,C,D,E,F can be solved using boundary conditions and Fourier Techniques (Fourier Series , Fourier Transforms)

4 Separation of Variable in Cylindrical Coordinate

To avoid the equations being too long , shorthand notations are used : e.g. $X(x)=X$, $\dfrac{dZ(z)}{dz}=Z'$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V(r,\phi,z)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V(r,\phi,z)}{\partial\phi^2}+\frac{\partial^2 V(r,\phi,z)}{\partial z^2}=0$$

Let $V=R\Phi Z$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R\Phi Z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 R\Phi Z}{\partial\phi^2}+\frac{\partial^2 R\Phi Z}{\partial z^2}=0$$

$$\frac{\Phi Z}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{RZ}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}+R\Phi\frac{\partial^2 Z}{\partial z^2}=0$$

Same, divide whole equation by $R\Phi Z$

$$\underbrace{\frac{1}{Rr}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{1}{r^2\Phi}\frac{\partial^2\Phi}{\partial\phi^2}}_{Z-independent}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=0$$

Expand , and using shorthand notation

$$\frac{R'}{Rr}+\frac{R"}{R}+\frac{\Phi"}{r^2\Phi}+\frac{Z"}{Z}=0$$

Let

$$\frac{Z"}{Z}=a^2 \qquad \Rightarrow Z(z)=\begin{cases} Ae^{az}+Be^{-az} \\ A\cosh az+B\sinh az \end{cases}$$

Thus the remaining part is

$$\frac{R'}{Rr}+\frac{R"}{R}+\frac{\Phi"}{r^2\Phi}+a^2=0$$

$$\frac{rR'}{R}+\frac{r^2R"}{R}+\frac{\Phi"}{\Phi}+r^2a^2=0$$

Let

$$\frac{\Phi"}{\Phi}=-b^2 \qquad \Rightarrow \Phi(\phi)=\begin{cases} Ce^{jb\phi}+De^{-jb\phi} \\ C\cos b\phi+D\sin b\phi \end{cases}$$

The remaining part is

$$\frac{rR'}{R}+\frac{r^2R"}{R}-b^2+r^2a^2=0$$

$$\frac{R'}{r}+R"+\left(a^2-\frac{b^2}{r^2}\right)R=0$$

Which is

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(a^2-\frac{b^2}{r^2}\right)R=0$$

This is the Bessel's Equation, the solution is $R=EJ(r)+FY(r)$, where J,Y are the Bessel's function of the first kind and second kind

Thus the final solution will be

$$V=\begin{cases} \left(Ae^{az}+Be^{-az}\right)\left(Ce^{jb\phi}+De^{-jb\phi}\right)\left(EJ(r)+FY(r)\right) \\ \left(A\cosh az+B\sinh az\right)\left(C\cos b\phi+D\sin b\phi\right)\left(EJ(r)+FY(r)\right) \end{cases}$$

(About how to solve Bessel's Function will be mentioned in another document)

5 Separation of Variables in Spherical Coordinate

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V(r,\phi,\theta)}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V(r,\phi,\theta)}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2V(r,\phi,\theta)}{\partial\phi^2}=0$$

The solution is selected that it depends on r,θ but not on ϕ

Let

$$V=R\Theta$$

$$\frac{\Theta}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=0$$

Expand

$$\frac{2\Theta}{r}\frac{\partial R}{\partial r}+\Theta\frac{\partial^2 R}{\partial r^2}+\frac{R\cot\theta}{r^2}\frac{\partial\Theta}{\partial\theta}+\frac{R}{r^2}\frac{\partial^2\Theta}{\partial\theta^2}=0$$

$$\frac{2r}{R}\frac{\partial R}{\partial r}+\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{\cot\theta}{\Theta}\frac{\partial\Theta}{\partial\theta}+\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2}=0$$

And specially set the constant to be $n(n+1)$

$$\frac{2r}{R}\frac{\partial R}{\partial r}+\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}=n(n+1)\qquad\frac{\cot\theta}{\Theta}\frac{\partial\Theta}{\partial\theta}+\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2}=-n(n+1)$$

$$\frac{\partial^2 R}{\partial r^2}+2r\frac{\partial R}{\partial r}-n(n+1)\frac{R}{r^2}=0\qquad\frac{\partial^2\Theta}{\partial\theta^2}+\cot\theta\frac{\partial\Theta}{\partial\theta}+n(n+1)\Theta=0$$

The first equation has the solution form as

$$R=Ar^n+Br^{-(n+1)}$$

The second one is the Legendre Equation, the solution is the Legendre polynomials.

$$P(\xi)=\frac{1}{2^n n!}\frac{d^n}{d\xi^n}\left(\xi^2-1\right)^n$$

Thus the final solution

$$V=R\Theta$$

(How to exactly solve the Legendre Eqautions will be mentioned in another document.)

$-END-$