Maxwell's Equations

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1 Maxwell's equations in integral form

1.1 Faradya's Law of induction

The induced emf (electromotive force) in a closed circuit is equal to the negative of rate of change of magnetic flux pass through it.

$$\oint_{\partial S} \bar{E} \cdot d\bar{l} = -\frac{\partial \Phi_B}{\partial t} = -\frac{\partial}{\partial t} \oiint_S \bar{B} \cdot d\bar{S}$$

1.2 Ampère's circuital law with Maxwell's Correction

The mmf (magnetomotive force) in a closed circuit is equal to the rate of change of electric flux and current pass through it.

$$\oint_{\partial S} \bar{H} \cdot d\bar{l} = \oiint_S \left(\bar{J} + \frac{\partial \bar{D}}{\partial t} \right) \cdot d\bar{S}$$

1.3 Gauss's Law for electricity

The electric flux pass through any closed surface is proportional to the enclosed electric charge.

$$\oiint_S \bar{D} \cdot d\bar{S} = \Phi_D = Q = \rho_v V$$

1.4 Gauss's Law for magnetism

The magnetic flux pass through any closed surface is zero

$$\oiint \bar{B} \cdot d\bar{S} = 0$$

2 Maxwell's equations in differential form

2.1 Vector Identities

2.1.1 Gauss's Divergence Theorem

$$\iiint_V \nabla \cdot \bar{V} dV = \oiint_{\partial V} \bar{V} \cdot d\bar{S}$$

2.1.2 Stokes' Curl Theorem

$$\iint_S \nabla \times \bar{V} \cdot d\bar{S} = \oint_{\partial S} \bar{V} \cdot d\bar{l}$$

2.2 Faraday's Law in differential form

Apply Stoke's Theorem

$$\oint_{\partial S} \bar{E} \cdot d\bar{l} = -\frac{\partial}{\partial t} \oiint_S \bar{B} \cdot d\bar{S} \qquad \longrightarrow \qquad \oiint_S \nabla \times \bar{E} \cdot d\bar{S} = -\frac{\partial}{\partial t} \oiint_S \bar{B} \cdot d\bar{S}$$

Rearrange

$$\oiint_S (\nabla \times \bar{E}) \cdot d\bar{S} = \oiint_S \left(-\frac{\partial \bar{B}}{\partial t}\right) \cdot d\bar{S}$$

The integral kernel thus equal to each other

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

2.3 Ampère's circuital law in differential form

Apply Stoke's Theorem

$$\oint_{\partial S} \bar{H} \cdot d\bar{l} = \oiint_S \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right) \cdot d\bar{S} \qquad \longrightarrow \qquad \oiint_S \nabla \times \bar{H} \cdot d\bar{S} = \oiint_S \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right) \cdot d\bar{S}$$

The integral kernel thus equal to each other

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

2.4 Gauss's Law for electricity in differential form

Apply Gauss's Divergence Theorem

$$\oiint_S \bar{D} \cdot d\bar{S} = Q \qquad \longrightarrow \qquad \iiint_V \nabla \cdot \bar{D} dV = Q = \iiint \rho_v dV$$

The integral kernel thus equal to each other

$$\nabla \cdot \bar{D} = \rho_v$$

2.5 Gauss's Law for magnetism in differential form

Apply Gauss's Divergence Theorem

$$\oiint_S \bar{B} \cdot d\bar{S} = 0 \qquad \longrightarrow \qquad \iiint_V \nabla \cdot \bar{B} dV = 0 = \iiint 0 dV$$

The integral kernel thus equal to each other

$$\nabla \cdot \bar{B} = 0$$

3 Conservation of electric charge

Consider the follow equations

$$\begin{cases} \nabla \times \bar{H} = \bar{J} + \dfrac{\partial \bar{D}}{\partial t} & \text{Faraday's Law} \\ \nabla \cdot \bar{D} = \rho_v & \text{Gauss's Law} \\ \nabla \cdot \left(\nabla \times \bar{V}\right) = 0 & \text{Div of curl is zero} \end{cases}$$

Take the div of Faraday's Law

$$\nabla \cdot \left(\nabla \times \bar{H}\right) = \nabla \cdot \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right)$$

$$0 = \nabla \cdot \bar{J} + \frac{\partial}{\partial t} \nabla \cdot \bar{D}$$

$$0 = \nabla \cdot \bar{J} + \frac{\partial \rho_v}{\partial t}$$

i.e. The rate of current transfer out of a volume eqaul to decreasing rate of charge in that volume

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

4 Maxwell's equations in static case

Static $\iff \dfrac{\partial}{\partial t} = 0$

$$\begin{matrix} \nabla \times \bar{E} = 0 \\ \nabla \times \bar{H} = J \\ \nabla \cdot \bar{B} = 0 \\ \nabla \cdot \bar{D} = \rho_v \\ \nabla \cdot \bar{J} = 0 \end{matrix} \qquad \begin{matrix} \begin{cases} \nabla \times \bar{E} = 0 \\ \nabla \cdot \bar{D} = \rho_v \end{cases} \\ \\ \begin{cases} \nabla \times \bar{H} = J \\ \nabla \cdot \bar{B} = 0 \\ \nabla \cdot \bar{J} = 0 \end{cases} \end{matrix}$$

E & D field are generated by ρ_v
H & B fields are generated by J
J & ρ_v are independent $\left(\nabla \cdot \bar{J} = 0\right)$
E , H are decoupled, the electrostatic field and magnetostatic field are independent.

5 Maxwell's Equations in source-free dynamic case

Source free $\iff J = \rho_v = 0$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$
$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t}$$
$$\nabla \cdot \bar{B} = 0$$
$$\nabla \cdot \bar{D} = 0$$

E & H are coupled, they are not independent.
This coupling generate the phenomenon of EM wave propagation

6 Maxwell's equations in phasor form

6.1 Phasor Review

Apply Euler's Eqaution to sinusoidal term

$$V(t) = V_0 \cos(\omega t + \phi) = \Re e\left[V_0 e^{j(\omega t + \phi)}\right] = \Re e\left[\left(V_0 e^{j\phi}\right) e^{j\omega t}\right]$$

The phasor form is thus

$$V_0 e^{j\phi}$$

i.e.

$$V(t) = V_0 \cos(\omega t + \phi) \quad \longleftrightarrow \quad \widetilde{V} = V_0 e^{j\phi}$$

- Original time-domain form is real number
- Phasor form is complex number

6.2 Phasor Differentiation and Integration

$$\frac{\partial}{\partial t} V(t) = \frac{\partial}{\partial t} V_0 \cos(\omega t + \phi) = -\omega V_0 \sin(\omega t + \phi) = \Re e\left[\left(j\omega V_0 e^{j\phi}\right) e^{j\omega t}\right]$$

Therefore

$$\frac{\partial}{\partial t} V(t) \longleftrightarrow j\omega \widetilde{V}$$

$$\int V(t)dt = \int V_0 \cos(\omega t + \phi)\, dt = \frac{1}{\omega} V_0 \sin(\omega t + \phi) = \Re e\left[\left(\frac{1}{j\omega} V_0 e^{j\phi}\right) e^{j\omega t}\right]$$

Therefore

$$\int V(t)dt \longleftrightarrow \frac{1}{j\omega} \widetilde{V}$$

6.3 Maxwell's Equations in Phasor Form

$$\nabla \times \bar{E} = -j\omega \bar{B}$$

$$\nabla \times \bar{H} = j\omega \bar{D} + \bar{J}$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \cdot \bar{D} = \rho_v$$

$$\nabla \cdot \bar{J} = -j\omega \rho_v$$

The equations are now complex and time-independent.

7 The dependency in Maxwell's Equations

There are 4 equations, but the other equations can be derived form the 2 curl equations with some vector identities.

$$\nabla \times \bar{E} = -j\omega \bar{B}$$

$$\nabla \times \bar{H} = j\omega \bar{D} + \bar{J}$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \cdot \bar{D} = \rho_v$$

$$\nabla \cdot \bar{J} = -j\omega \rho_v$$

7.1 Conservation of charge derived from Ampere's Law

Already shown previously

7.2 Gauss's Law for electricity derived from Ampere's law & Conservation of Charge

Apply div of curl is zero into Ampere's Law

$$\nabla \times \bar{H} = j\omega \bar{D} + \bar{J} \qquad \longrightarrow \qquad \nabla \cdot \left(\nabla \times \bar{H}\right) = \nabla \cdot \left(j\omega \bar{D} + \bar{J}\right)$$

$$0 = j\omega \nabla \cdot \bar{D} + \nabla \cdot \bar{J}$$

Apply Conservation of charge $\nabla \cdot \bar{J} = -j\omega \rho_v$

$$0 = j\omega \nabla \cdot \bar{D} - j\omega \rho_v$$

i.e.

$$\nabla \cdot \bar{D} = \rho_v$$

7.3 Gauss's Law for magnetism derived from Faraday's Law

Apply div of curl is zero into Faraday's Law

$$\nabla \times \bar{E} = j\omega \bar{B} \qquad \longrightarrow \qquad \nabla \cdot \big(\nabla \times \bar{E}\big) = \nabla \cdot \big(j\omega \bar{B}\big)$$

$$0 = \nabla \cdot \big(j\omega \bar{B}\big) = j\omega \nabla \cdot \bar{B}$$

i.e.

$$\nabla \cdot \bar{B} = 0$$