Electromagnetic Waves I

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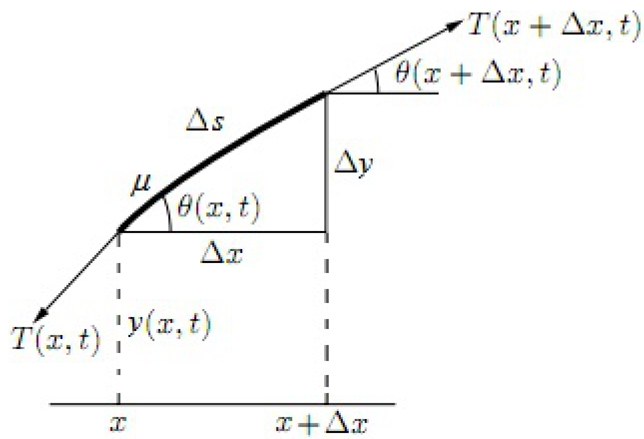
1 Derivation of Wave Equation of string



For one Dimensional Wave

$$Y = y(x,t)$$

The net upward force is

$$T(x+\Delta x,t) - T(x,t) = T\sin\theta_{x+\Delta x} - T\sin\theta_x = T\left(\sin\theta_{x+\Delta x} - \sin\theta_x\right)$$

For a small vibration,

$$\Delta x \to 0 \implies \theta \to 0 \iff \sin\theta \simeq \tan\theta \simeq \theta$$

Also ,

$$\tan\theta = \lim_{\Delta x \to 0} \frac{\Delta y(x,t)}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x,t) - y(x,t)}{\Delta x} = \frac{\partial y}{\partial x}$$

Therefore ,

$$\sin\theta \simeq \frac{\partial y}{\partial x}$$

Hence , the net upward force is

$$T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right)$$

By Newton's Second Law, net external force = mass×acceleration = length× linear mass density× acceleration

$$T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) = F = \Delta s \cdot \mu \cdot \left(\frac{\partial^2 y}{\partial t^2} + \epsilon\right)$$

As the vibration is small , $\Delta s \simeq \Delta x$

$$T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) = \mu \Delta x \left(\frac{\partial^2 y}{\partial t^2} + \epsilon\right)$$

Re-arrange the terms

$$\frac{T}{\mu} \frac{\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}}{\Delta x} = \frac{\partial^2 y}{\partial t^2} + \epsilon$$

Take the limit

$$LHS \quad : \quad \lim_{\Delta x \to 0} \frac{\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}}{\Delta x} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)$$

$$RHS \quad : \quad \lim_{\Delta x \to 0} \epsilon = 0$$

Thus

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y(x,t)}{\partial t^2} \qquad c = \sqrt{\frac{T}{\mu}}$$

For the 3D case

$$U = u(x,y,z,t)$$

The Generalized Wave Equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

In short

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

2 Derivation of the EM wave Equation in time domain

Consider the time domain Maxwell Eqaution in a source free region in the media characteriszed by $(\mu, \varepsilon, \sigma)$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \cdot \mathbf{E} = 0$$
$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \cdot \mathbf{H} = 0$$

Recall of vector identity

$$\nabla \times \nabla \times \bar{V} = \nabla \left(\nabla \cdot \bar{V} \right) - \nabla^2 \bar{V}$$

Take the curl of Faraday's Law, and consider the left hand side

$$\nabla \times \nabla \times \mathbf{E} = \nabla \left(\nabla \cdot \mathbf{E} \right) - \nabla^2 \mathbf{E}$$

Plug in the Gauss's Law

$$\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E}$$

Then consider the right hand side

$$\nabla \times \nabla \times \mathbf{E} = \nabla \times \left(-\mu \frac{\partial \mathbf{H}}{\partial t} \right) = -\mu \frac{\partial}{\partial t} \left(\nabla \times \mathbf{H} \right)$$

Plugin the Ampere Law

$$\nabla \times \nabla \times \mathbf{E} = -\varepsilon\mu \frac{\partial}{\partial t} \frac{\partial \mathbf{E}}{\partial t} = -\varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Equate the left hand side and right hand side

$$-\nabla^2 \mathbf{E} = \nabla \times \nabla \times \mathbf{E} = -\varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Therefore

$$\nabla^2 \mathbf{E} - \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

In the same way,

$$\nabla^2 \mathbf{H} - \varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

Compare the EM wave equation to the general wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

The propagation velocity of the wave is thus

$$c = \frac{1}{\sqrt{\mu\varepsilon}}$$

In free space

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \approx 3 \times 10^8 ms^{-1}$$

3 Derivation of the EM Wave Equation in phasor domain

Consider the Phasor form Maxwell Equation in the media characterized by $(\mu, \varepsilon, \sigma)$

$$\begin{aligned} \nabla \times \widetilde{E} &= -j\omega\mu\widetilde{H} \qquad & \nabla \cdot \widetilde{E} = 0 \\ \nabla \times \widetilde{H} &= \sigma\widetilde{E} + j\omega\varepsilon\widetilde{E} \qquad & \nabla \cdot \widetilde{H} = 0 \end{aligned}$$

Recall of vector identity

$$\nabla \times \nabla \times \bar{V} = \nabla \left(\nabla \cdot \bar{V}\right) - \nabla^2 \bar{V}$$

Take the curl of Faraday's Law, and consider the left hand side

$$\nabla \times \nabla \times \widetilde{E} = \nabla \left(\nabla \cdot \widetilde{E}\right) - \nabla^2 \widetilde{E}$$

Plug in the Gauss's Law

$$\nabla \times \nabla \times \widetilde{E} = -\nabla^2 \widetilde{E}$$

Then consider the right hand side

$$\nabla \times \nabla \times \widetilde{E} = \nabla \times \left(-j\omega\mu\widetilde{H}\right) = -j\omega\mu\nabla \times \widetilde{H}$$

Plugin the Ampere Law

$$\nabla \times \nabla \times \widetilde{E} = -j\omega\mu \left(\sigma\widetilde{E} + j\omega\varepsilon\widetilde{E}\right) = -j\omega\mu \left(\sigma + j\omega\varepsilon\right) \widetilde{E}$$

Equate the left hand side and right hand side

$$-\nabla^2 \widetilde{E} = \nabla \times \nabla \times \widetilde{E} = -j\omega\mu \left(\sigma + j\omega\varepsilon\right) \widetilde{E}$$

Therefore

$$\nabla^2 \widetilde{E} - j\omega\mu \left(\sigma + j\omega\varepsilon\right) \widetilde{E} = 0$$

Which is a wave eqaution.

Apply the same method to get the wave equation of $\widetilde{H}$
First, take the curl of the Ampere Law, and consider the left hand side

$$\nabla \times \nabla \times \widetilde{H} = \nabla \left(\nabla \cdot \widetilde{H}\right) - \nabla^2 \widetilde{H}$$

Apply Gauss Law of zero divergence

$$\nabla \times \nabla \times \widetilde{H} = -\nabla^2 \widetilde{H}$$

Then consider the right hand side

$$\nabla \times \nabla \times \widetilde{H} = (\sigma + j\omega\varepsilon) \nabla \times \widetilde{E}$$

Plug in the Faraday's Law

$$\nabla \times \nabla \times \widetilde{H} = -j\omega\mu (\sigma + j\omega\varepsilon) \widetilde{H}$$

Equate the left hand side and right hand side

$$-\nabla^2 \widetilde{H} = \nabla \times \nabla \times \widetilde{H} = -j\omega\mu (\sigma + j\omega\varepsilon) \widetilde{H}$$

Therefore

$$\nabla^2 \widetilde{H} - j\omega\mu (\sigma + j\omega\varepsilon) \widetilde{H} = 0$$

i.e. the 2 wave eqautions are

$$\nabla^2 \widetilde{E} - j\omega\mu (\sigma + j\omega\varepsilon) \widetilde{E} = 0 \qquad \nabla^2 \widetilde{H} - j\omega\mu (\sigma + j\omega\varepsilon) \widetilde{H} = 0$$

Let

$$j\omega\mu (\sigma + j\omega\varepsilon) = \gamma^2 \qquad \gamma : \text{the complex propagation constant}$$

The wave equations can be written in more compact form, the $phasor\ form\ vector\ Helmholtz\ eqaution$

$$\nabla^2 \widetilde{E} - \gamma^2 \widetilde{E} = 0 \qquad \nabla^2 \widetilde{H} - \gamma^2 \widetilde{H} = 0$$

4 The complex propagation constant

4.1 The 2 terms

$$\gamma = \sqrt{j\omega\mu (\sigma + j\omega\varepsilon)} = \alpha + j\beta$$

Where

γ complex propagation constant α attenuation constant β progatation constant

γ is the complex propagation constant. Unit m^{-1}

α is rate of decay as the wave propagate in lossy medium. (For lossless medium, $\alpha = 0$). Unit Np/m

β is the propagation constant. This is a very important constant, phase velocity, wavelength, frequency, can be derived using β. Unit rad/m

4.2 Expressing $\sqrt{j\omega\mu (\sigma + j\omega\varepsilon)}$ as standard complex number $\alpha + j\beta$

Using the following equation, α , β can be evaluated

$$\gamma = \sqrt{a + jb} = \sqrt{\frac{r + a}{2}} + j \,\mathrm{Sgn}(b) \sqrt{\frac{r - a}{2}} \qquad \text{where } r = \sqrt{a^2 + b^2}$$

Proof. Using Polar Form

For $z = a + jb$

$$z = |z|\angle z = r \exp\left(j \tan^{-1}\frac{b}{a}\right)$$

$$\gamma = \sqrt{z} = \sqrt{r}\exp\left[\frac{j}{2}\tan^{-1}\frac{b}{a}\right] = \sqrt{r}\cos\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right] + j\sqrt{r}\sin\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right]$$

The sign of b is important , since $\tan^{-1}\left(\frac{b}{a}\right)$ give out a angle, and $\sin(\pm\theta) = \pm\sin\theta$:

$$\sqrt{z} = \begin{cases} \sqrt{r}\cos\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right] + j\sqrt{r}\sin\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right] & b > 0 \\ \sqrt{r}\cos\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right] - j\sqrt{r}\sin\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right] & b < 0 \end{cases}$$

Using Sgn notation on b to combine the 2 equation

$$\sqrt{z} = \sqrt{r}\cos\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right] + j\mathrm{Sgn}(b)\sqrt{r}\sin\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right]$$

Turn sin into cos

$$= \sqrt{r}\cos\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right] + j\mathrm{Sgn}(b)\sqrt{r}\sqrt{1 - \cos^2\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right]}$$

To get rid of $\frac{1}{2}$: recall the Double Angle Formula : $\cos 2A = 2\cos^2 A - 1 \iff \cos A = \sqrt{\frac{\cos 2A + 1}{2}}$

$$= \sqrt{r}\sqrt{\frac{\cos\left(\tan^{-1}\frac{b}{a}\right) + 1}{2}} + j\mathrm{Sgn}(b)\sqrt{r}\sqrt{1 - \frac{\cos\left(\tan^{-1}\frac{b}{a}\right) + 1}{2}}$$

$$= \sqrt{r}\sqrt{\frac{\cos\left(\tan^{-1}\frac{b}{a}\right) + 1}{2}} + j\mathrm{Sgn}(b)\sqrt{r}\sqrt{\frac{\cos\left(\tan^{-1}\frac{b}{a}\right) - 1}{2}}$$

For $\tan\theta = \frac{b}{a}$, $\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{a}{\sqrt{a^2 + b^2}} = \frac{a}{r}$

$$= \sqrt{r}\sqrt{\frac{\frac{a}{r} + 1}{2}} + j\mathrm{Sgn}(b)\sqrt{r}\sqrt{\frac{\frac{a}{r} - 1}{2}} = \sqrt{\frac{a + r}{2}} + j\mathrm{Sgn}(b)\sqrt{\frac{a - r}{2}}$$

$\therefore$

$$\gamma = \sqrt{a + jb} = \sqrt{\frac{r + a}{2}} + j\,\mathrm{Sgn}(b)\sqrt{\frac{r - a}{2}}$$

□

Thus

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \sqrt{-\omega^2\mu\varepsilon + j\sigma\omega\mu}$$

$$
\begin{cases}
a = -\omega^2\mu\varepsilon \\
b = \sigma\omega\mu \in \mathbb{R}^+ \ \Rightarrow \text{sgn} b = +1 \\
r = \sqrt{a^2+b^2} = \omega\mu\sqrt{\sigma^2+\omega^2\varepsilon^2} = \omega^2\mu\varepsilon\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} \\
r+a = \omega^2\mu\varepsilon\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - \omega^2\mu\varepsilon = \omega^2\mu\varepsilon\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right] \\
r-a = \omega^2\mu\varepsilon\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + \omega^2\mu\varepsilon = \omega^2\mu\varepsilon\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]
\end{cases}
\Rightarrow
\begin{cases}
\alpha = Re\left(\gamma\right) = \sqrt{\frac{r+a}{2}} \\
\beta = Im\left(\gamma\right) = \text{sgn}\left(b\right)\sqrt{\frac{r-a}{2}}
\end{cases}
$$

i.e.

$$
\gamma = \sqrt{j\omega\mu\left(\sigma+j\omega\epsilon\right)} = \alpha + j\beta \qquad
\begin{aligned}
\alpha &= \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} \\
\beta &= \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}
\end{aligned}
$$

5 The Laplacian, the six component equations

Reconsider the wave equation / phasor form vector Helmholtz equations

$$
\nabla^2\widetilde{\mathbf{E}} - \gamma^2\widetilde{\mathbf{E}} = 0 \qquad \nabla^2\widetilde{\mathbf{H}} - \gamma^2\widetilde{\mathbf{H}} = 0
$$

the equations include the terms of vector laplacian $\nabla^2\widetilde{\mathbf{E}}$, $\nabla^2\widetilde{\mathbf{H}}$

Review of Laplacian operator in rectangular coordinate

$$
\nabla^2 = \nabla\cdot\nabla = \left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\cdot\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right) = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}
$$

Laplacian on scalar field $\mathit{\Psi}(x,y,z)$

$$
\nabla^2\mathit{\Psi} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\mathit{\Psi} = \frac{\partial^2\mathit{\Psi}}{\partial x^2}+\frac{\partial^2\mathit{\Psi}}{\partial y^2}+\frac{\partial^2\mathit{\Psi}}{\partial z^2} \in \text{Scalar}
$$

Laplacian on vector field $\bar{\mathbf{V}}(x,y,z)$

$$
\nabla^2\bar{\mathbf{V}} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(V_x\hat{x}+V_y\hat{y}+V_z\hat{z}\right)
$$

$$
= \left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}\right)\hat{x} + \left(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}+\frac{\partial^2 V_y}{\partial z^2}\right)\hat{y} + \left(\frac{\partial^2 V_z}{\partial x^2}+\frac{\partial^2 V_z}{\partial y^2}+\frac{\partial^2 V_z}{\partial z^2}\right)\hat{z}
$$

Therefore, for

$$
\nabla^2\widetilde{\mathbf{E}} - \gamma^2\widetilde{\mathbf{E}} = 0 \qquad \nabla^2\widetilde{\mathbf{H}} - \gamma^2\widetilde{\mathbf{H}} = 0
$$

Rearrange

$$\nabla^2\widetilde{\mathbf{E}} = \gamma^2\widetilde{\mathbf{E}} \qquad \nabla^2\widetilde{\mathbf{H}} = \gamma^2\widetilde{\mathbf{H}}$$

Expand the **E** and **H** into component form

$$\nabla^2\widetilde{E}_x\hat{x} + \nabla^2\widetilde{E}_x\hat{y} + \nabla^2\widetilde{E}_x\hat{z} = \gamma^2\widetilde{E}_x + \gamma^2\widetilde{E}_y + \gamma^2\widetilde{E}_z$$

$$\nabla^2\widetilde{H}_x\hat{x} + \nabla^2\widetilde{H}_x\hat{y} + \nabla^2\widetilde{H}_x\hat{z} = \gamma^2\widetilde{H}_x + \gamma^2\widetilde{H}_y + \gamma^2\widetilde{H}_z$$

Expand the Laplacian

$$\frac{\partial^2\widetilde{E}_x}{\partial x^2} + \frac{\partial^2\widetilde{E}_x}{\partial y^2} + \frac{\partial^2\widetilde{E}_x}{\partial z^2} = \gamma^2\widetilde{E}_x \qquad \frac{\partial^2\widetilde{H}_x}{\partial x^2} + \frac{\partial^2\widetilde{H}_x}{\partial y^2} + \frac{\partial^2\widetilde{H}_x}{\partial z^2} = \gamma^2\widetilde{H}_x$$

$$\frac{\partial^2\widetilde{E}_y}{\partial x^2} + \frac{\partial^2\widetilde{E}_y}{\partial y^2} + \frac{\partial^2\widetilde{E}_y}{\partial z^2} = \gamma^2\widetilde{E}_y \qquad \frac{\partial^2\widetilde{H}_y}{\partial x^2} + \frac{\partial^2\widetilde{H}_y}{\partial y^2} + \frac{\partial^2\widetilde{H}_y}{\partial z^2} = \gamma^2\widetilde{H}_y$$

$$\frac{\partial^2\widetilde{E}_z}{\partial x^2} + \frac{\partial^2\widetilde{E}_z}{\partial y^2} + \frac{\partial^2\widetilde{E}_z}{\partial z^2} = \gamma^2\widetilde{E}_z \qquad \frac{\partial^2\widetilde{H}_z}{\partial x^2} + \frac{\partial^2\widetilde{H}_z}{\partial y^2} + \frac{\partial^2\widetilde{H}_z}{\partial z^2} = \gamma^2\widetilde{H}_z$$

These six equations contains all the information of all the components of the propagating EM wave in phasor form (in rectangular coordinate)

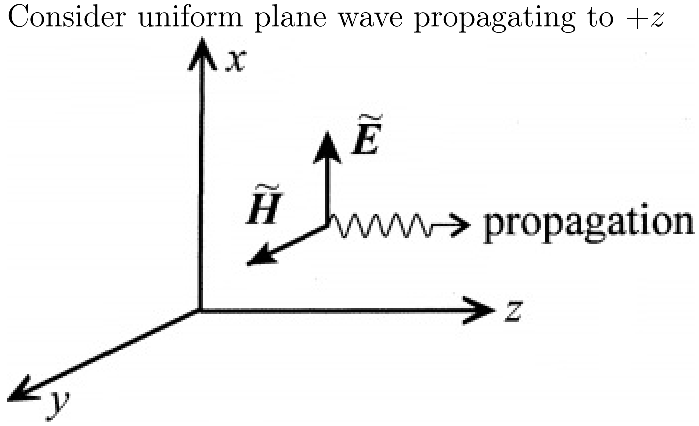
6 Uniform Plane Wave

Not all EM wave will contains all the six component.

For plane wave : **E** , **H** coplanar , and **E** $\perp$ **H** $\perp$ propagation difection

For uniform plane wave : **E**, **H** is same on in the plane $\perp$to propagation direction, it only varies in the direction of propagation

Consider uniform plane wave propagating to $+z$



E has only x component : $\widetilde{E}_y = \widetilde{E}_z = 0$

H has only y component : $\widetilde{H}_x = \widetilde{H}_z = 0$

So the six component eqautions are reduced into

$$\frac{\partial^2\widetilde{E}_x(x,y,z)}{\partial x^2} + \frac{\partial^2\widetilde{E}_x(x,y,z)}{\partial y^2} + \frac{\partial^2\widetilde{E}_x(x,y,z)}{\partial z^2} = \gamma^2\widetilde{E}_x(x,y,z)$$

$$\frac{\partial^2\widetilde{H}_y(x,y,z)}{\partial x^2} + \frac{\partial^2\widetilde{H}_y(x,y,z)}{\partial y^2} + \frac{\partial^2\widetilde{H}_y(x,y,z)}{\partial z^2} = \gamma^2\widetilde{H}_y(x,y,z)$$

Since it is uniform wave, the variation of magnitude only occur along the propagation direction z , so $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial y} = 0$

The 2 equations are further simplified as

$$\frac{\partial^2 \widetilde{E}_x(z)}{\partial z^2} = \gamma^2 \widetilde{E}_x(z) \qquad \frac{\partial^2 \widetilde{H}_y(z)}{\partial z^2} = \gamma^2 \widetilde{H}_y(z)$$

Finally, the PDE can be reduced into ODE

$$\frac{d^2 \widetilde{E}_x(z)}{dz^2} = \gamma^2 \widetilde{E}_x(z) \qquad \frac{d^2 \widetilde{H}_y(z)}{dz^2} = \gamma^2 \widetilde{H}_y(z)$$

With the general solution as

$$\widetilde{E}_x(z) = E_1 e^{\gamma z} + E_2 e^{-\gamma z} = E_1 e^{\alpha z} e^{j\beta z} + E_2 e^{-\alpha z} e^{-j\beta z}$$

$$\widetilde{H}_x(z) = H_1 e^{\gamma z} + H_2 e^{-\gamma z} = H_1 e^{\alpha z} e^{j\beta z} + H_2 e^{-\alpha z} e^{-j\beta z}$$

Remark. E_1, E_2 , H_1 , H_2 are all complex number , $E_1 = |E_1| e^{j\theta_{E1}}$

Change the equation from phasor back to time domain, consider the component

$$E_x(z,t) = \Re e \left\{ \widetilde{E}_x(z) e^{j\omega t} \right\} = \Re e \left\{ \left(E_1 e^{\alpha z} e^{j\beta z} + E_2 e^{-\alpha z} e^{-j\beta z} \right) e^{j\omega t} \right\}$$

$$= \Re e \left\{ |E_1| e^{j\theta_{E1}} e^{\alpha z} e^{j(\beta z + \omega t)} + |E_2| e^{j\theta_{E2}} e^{-\alpha z} e^{j(\omega t - \beta z)} \right\} = \Re e \left\{ |E_1| e^{\alpha z} e^{j(\omega t + \beta z + \theta_{E1})} + |E_2| e^{-\alpha z} e^{j(\omega t - \beta z + \theta_{E2})} \right\}$$

$$E_x(z,t) = |E_1| e^{\alpha z} \cos(\omega t + \beta z + \theta_{E1}) + |E_2| e^{-\alpha z} \cos(\omega t - \beta z + \theta_{E1})$$

In same logic

$$H_x(z,t) = |H_1| e^{\alpha z} \cos(\omega t + \beta z + \theta_{H1}) + |H_2| e^{-\alpha z} \cos(\omega t - \beta z + \theta_{H1})$$

Thus, the wave equations for uniform plane wave are

$$\mathbf{E}(z,t) = E_x(z,t)\hat{z} = \left[|E_1| e^{\alpha z} \cos(\omega t + \beta z + \theta_{E1}) + |E_2| e^{-\alpha z} \cos(\omega t - \beta z + \theta_{E1}) \right] \hat{z}$$

$$\mathbf{H}(z,t) = H_x(z,t)\hat{z} = \left[|H_1| e^{\alpha z} \cos(\omega t + \beta z + \theta_{H1}) + |H_2| e^{-\alpha z} \cos(\omega t - \beta z + \theta_{H1}) \right] \hat{z}$$

Consider the E-field, it consists of 2 terms

$$E_x(z,t) = |E_1| e^{\alpha z} \cos(\omega t + \beta z + \theta_{E1}) + |E_2| e^{-\alpha z} \cos(\omega t - \beta z + \theta_{E1})$$

1st term magnitude	$\lvert E_1\rvert e^{\alpha z}$	1st term phase	$\omega t + \beta z + \theta_{E1}$
2nd term magnitude	$\lvert E_2\rvert e^{-\alpha z}$	2nd terms phase	$\omega t - \beta z + \theta_{E1}$

Consider the phase

$$\omega t \pm \beta z + \theta = \text{constant}$$

When t increase (time move) , inorder to have a constant phase, the z should increase / decrease

$\omega t + \beta z + \theta_{E1} = \text{constant}$	time increase	z should decrease = move backward
$\omega t - \beta z + \theta_{E1} = \text{constant}$	time increase	z should increase = move forward

Thus

$$E_x(z,t) = \underbrace{|E_1|e^{\alpha z}\cos(\omega t + \beta z + \theta_{E1})}_{Backward\ wave} + \underbrace{|E_2|e^{-\alpha z}\cos(\omega t - \beta z + \theta_{E1})}_{Forward\ wave}$$

Consider the phase

$$\omega t \pm \beta z + \theta = \text{constant}$$

$$z = \frac{\pm 1}{\beta}(\text{constant} - \omega t - \theta)$$

The *phase velocity* is hence

$$\mathbf{v}_p = \frac{dz}{dt}\hat{z} = \frac{\pm\omega}{\beta}\hat{z}$$

Thus, the magnitude of phase velocity is

$$v_p = \frac{\omega}{\beta}$$

And thus, the *wavelength* is

$$\lambda = \frac{v_p}{f} = \frac{\omega/\beta}{f} = \frac{2\pi f/\beta}{f} = \frac{2\pi}{\beta}$$

Finally, the *intrisic impedance* is defined as $\dfrac{E}{H}$ since E is related to V and H is related to I
Consider fowrad wave of the phasor form equation

$$\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_x + \widetilde{\mathbf{E}}_y + \widetilde{\mathbf{E}}_z = E_0 e^{-\gamma z}\hat{x} + 0\hat{y} + 0\hat{z}$$

where E_0 is real, not complex (no phase)
Apply the Faraday's Law

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$$

$$\widetilde{\mathbf{H}} = -\frac{1}{j\omega\mu}\nabla \times \widetilde{\mathbf{E}} = -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_0 e^{-\gamma z} & 0 & 0 \end{vmatrix}$$

- Since $\widetilde{\mathbf{E}}$ has only $x-$component, so $\widetilde{\mathbf{E}}_y = \widetilde{\mathbf{E}}_z = 0$
- Since $\widetilde{\mathbf{E}}$ is only z-dependent , $\widetilde{\mathbf{E}}(z)$, so $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial y} = 0$

$$\widetilde{\mathbf{H}} = -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ E_0 e^{-\gamma z} & 0 & 0 \end{vmatrix} = -\frac{1}{j\omega\mu}\begin{vmatrix} 0 & \dfrac{\partial}{\partial z} \\ E_0 e^{-\gamma z} & 0 \end{vmatrix}(-\hat{y}) = -\frac{1}{j\omega\mu}\left(-\frac{\partial E_0 e^{-\gamma z}}{\partial z}\right)(-\hat{y})$$

$$= \frac{\gamma}{j\omega\mu}E_0 e^{-\gamma z}\hat{y} = H_0 e^{-\gamma z}\hat{y} \qquad \text{where } H_0 = \frac{\gamma}{j\omega\mu}E_0$$

- $H_0 = -\dfrac{\gamma}{j\omega\mu}E_0$ means the magnitude of H-field and E-field *inphase*

The *intrinsic impedance* is defined as

$$\eta = \frac{E_0}{H_0} = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu\left(\sigma + j\omega\varepsilon\right)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

Remark. In free space, $\sigma = 0$ (no conductivity)

$$\eta_0 = \sqrt{\frac{j\omega\mu_0}{j\omega\varepsilon_0}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 120\pi\,\Omega$$

Express the η in polar form

Becuase of the η_0 , it is better to extract $\sqrt{\dfrac{\mu}{\varepsilon}}$ out

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{j\omega}{\frac{\sigma}{\varepsilon} + j\omega}}$$

For rationalization of the denominator, extract the ω term out in the denominator

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{j\omega}{\frac{\sigma}{\varepsilon} + j\omega}} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{j\omega}{j\omega\left(\frac{\sigma}{j\varepsilon\omega} + 1\right)}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1 + \frac{\sigma}{j\varepsilon\omega}}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1 - j\frac{\sigma}{\varepsilon\omega}}}$$

For the term $\sqrt{1 - j\frac{\sigma}{\varepsilon\omega}}$

$$\frac{1}{\sqrt{1 - j\frac{\sigma}{\varepsilon\omega}}} = \left(1 - j\frac{\sigma}{\varepsilon\omega}\right)^{-\frac{1}{2}}$$

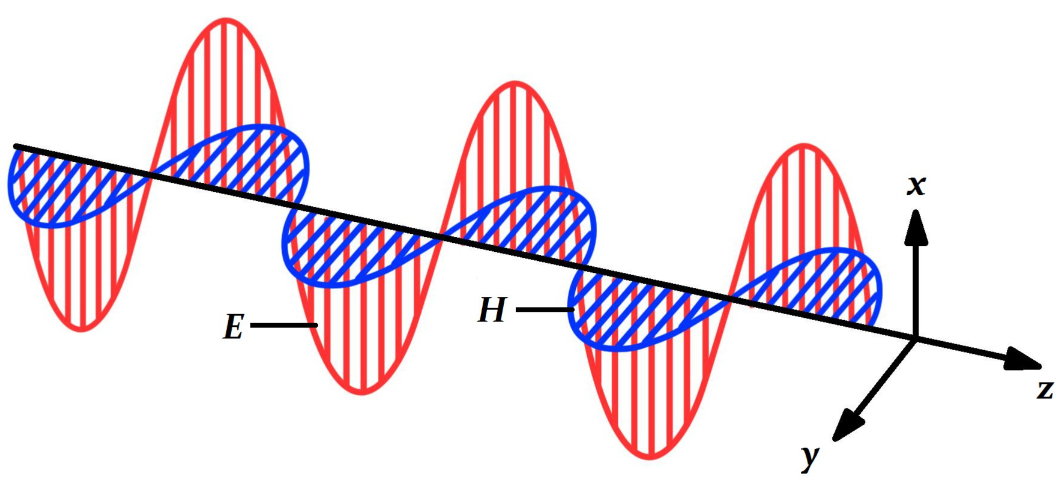
In polar form

$$\left(1 - j\frac{\sigma}{\varepsilon\omega}\right)^{-\frac{1}{2}} = \sqrt{\left(1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right)^{-\frac{1}{2}}} \exp\left(j\frac{1}{2}\tan^{-1}\frac{\sigma}{\varepsilon\omega}\right)$$

Therefore

$$\eta = |\eta|e^{j\theta_\eta} = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{\frac{1}{4}}} \exp\left(j\frac{1}{2}\tan^{-1}\frac{\sigma}{\varepsilon\omega}\right)$$

Small summary of properties of uniform plane EM wave



- $\mathbf{E} \perp \mathbf{H}$
- $\mathbf{E} \times \mathbf{H} =$ propagation direction , and $\mathbf{E} \perp \mathbf{H} \perp$ propagation direction
- $\mathbf{E}$, $\mathbf{H}$ magnitude is same for the plane, it only change along propagation direction
- For lossless media, E and H inphase : when E is max, H is also max, when E is min, H is also min (the diagram above)
- For lossy media, E and H are not inphase : phase angle differ by θ_η
- The solution of wave equations consist of both forward wave and backward wave
- $\widetilde{\mathbf{E}} = \eta \widetilde{\mathbf{H}} \times \hat{\mathbf{k}}$, $\widetilde{\mathbf{H}} = \frac{1}{\eta} \widetilde{\mathbf{k}} \times \widetilde{\mathbf{E}}$, where $\hat{\mathbf{k}}$ is wave vector that in direction of propagation

7 The parameters of EM waves

7.1 General parameters in lossy medium ($\sigma > 0$, $\mu = \mu_r \mu_0$, $\varepsilon = \varepsilon_r \varepsilon_0$)

$$\gamma = \sqrt{j\omega\mu\left(\sigma + j\omega\varepsilon\right)}$$

$$\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} \qquad \beta = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{4\pi}{\sqrt{\mu\varepsilon\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{\frac{1}{4}}} \exp\left(j\frac{1}{2}\tan^{-1}\frac{\sigma}{\varepsilon\omega}\right)$$

$$\widetilde{\mathbf{E}}(z) = E_0 e^{-\gamma z}\hat{x} \qquad \mathbf{E}(z,t) = E_0 e^{-\alpha z}\cos\left(\omega t - \beta z\right)\hat{x}$$

$$\widetilde{\mathbf{H}}(z) = \eta E_0 e^{-\gamma z}\hat{y} \qquad \mathbf{H}(z,t) = |\eta| E_0 e^{-\alpha z}\cos\left(\omega t - \beta z + \theta_\eta\right)\hat{y}$$

7.2 For lossless media ($\sigma = 0$, $\mu = \mu_r \mu_0, \varepsilon = \varepsilon_r \varepsilon_0$)

$$\gamma = \sqrt{-\omega^2 \mu\epsilon} = j\omega\sqrt{\mu\varepsilon} \qquad \text{pure imaginary number}$$

$$\alpha = 0 \qquad \beta = \omega\sqrt{\mu\varepsilon}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \text{ real, no angle}$$

$$\widetilde{\mathbf{E}}(z) = E_0 e^{-\beta z}\hat{x} \qquad \mathbf{E}(z,t) = E_0 \cos(\omega t - \beta z)\,\hat{x}$$

$$\widetilde{\mathbf{H}}(z) = \sqrt{\frac{\mu}{\varepsilon}} E_0 e^{-\beta z}\hat{y} \qquad \mathbf{H}(z,t) = \sqrt{\frac{\mu}{\varepsilon}} E_0 \cos(\omega t - \beta z)\,\hat{y}$$

7.3 In free space $(\sigma = 0\,,\ \mu_0\,,\ \varepsilon_0)$

$$\gamma = j\omega\sqrt{\mu_0\varepsilon_0}$$

$$\alpha = 0 \qquad \beta = \omega\sqrt{\mu_0\varepsilon_0}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu_0\varepsilon_0}} = \frac{c}{f}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 120\pi$$

$$\widetilde{\mathbf{E}}(z) = E_0 e^{-\beta z}\hat{x} \qquad \mathbf{E}(z,t) = E_0 \cos(\omega t - \beta z)\,\hat{x}$$

$$\widetilde{\mathbf{H}}(z) = 120\pi E_0 e^{-\beta z}\hat{y} \qquad \mathbf{H}(z,t) = 120\pi E_0 \cos(\omega t - \beta z)\,\hat{y}$$

7.4 Perfect conductor $(\sigma = \infty\,,\ \mu = \mu_r\mu_0\,, \varepsilon = \varepsilon_r\varepsilon_0\,)$

$$\gamma = \infty$$

$$\alpha = \infty \qquad \beta = \infty$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\infty} = 0$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\infty} = 0$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{\infty}} = 0$$

$$\widetilde{\mathbf{E}}(z) = E_0 e^{-\infty z}\hat{x} = 0 \qquad \mathbf{E}(z,t) = E_0 e^{-\infty z} \cos(\omega t - \beta z)\,\hat{x} = 0$$

$$\widetilde{\mathbf{H}}(z) = (0) E_0 e^{-\infty z}\hat{y} = 0 \qquad \mathbf{H}(z,t) = (0) E_0 e^{-\infty z} \cos(\omega t - \beta z + \theta_\eta)\,\hat{y} = 0$$

Remark. It means no field propagating inside perfect conductor

7.5 Good conductor (large σ : $\dfrac{\sigma}{\omega\varepsilon} \gg 1$, $\mu = \mu_r\mu_0$, $\varepsilon = \varepsilon_r\varepsilon_0$)

$$\gamma = \sqrt{j\omega\mu\left(\sigma + j\omega\varepsilon\right)}$$

$$\alpha = \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} \mp 1\right]} \approx \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{\left(\frac{\sigma}{\omega\varepsilon}\right)^2} \mp 1\right]} = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\frac{\sigma}{\omega\varepsilon} \mp 1\right]} \approx \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\frac{\sigma}{\omega\varepsilon}\right]}$$

$$= \omega\sqrt{\frac{\mu}{2}\frac{\sigma}{\omega}} = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}$$

$$v_p = \frac{\omega}{\beta} \approx \frac{\omega}{\sqrt{\pi f\mu\sigma}} = \sqrt{\frac{4\pi f}{\mu\sigma}}$$

$$\lambda = \frac{2\pi}{\beta} \approx \frac{2\pi}{\sqrt{\pi f\mu\sigma}} = \sqrt{\frac{4\pi}{f\mu\sigma}}$$

$$\eta = \frac{\sqrt{\mu/\varepsilon}}{\left[1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{\frac{1}{4}}}\exp\left(j\frac{1}{2}\tan^{-1}\frac{\sigma}{\varepsilon\omega}\right) \approx \frac{\sqrt{\mu/\varepsilon}}{\left[\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{\frac{1}{4}}}\exp\left(j\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{\frac{\sigma}{\varepsilon\omega}}}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$$

$$= \sqrt{\frac{\mu\omega}{\sigma}}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{\pi f\mu}{\sigma}}\left(1+j\right) = \sqrt{\frac{2\pi f\mu}{\sigma}}e^{j\frac{\pi}{4}}$$

$$\widetilde{\mathbf{E}}(z) = E_0 e^{-\gamma z}\hat{x} \qquad \mathbf{E}(z,t) \approx E_0\exp\left[-\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\frac{\sigma}{\omega\varepsilon}\right]}z\right]\cos\left(\omega t - \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\frac{\sigma}{\omega\varepsilon}\right]}z\right)\hat{x}$$

$$\widetilde{\mathbf{H}}(z) \approx \sqrt{\frac{2\pi f\mu}{\sigma}}e^{j\frac{\pi}{4}}E_0 e^{-\gamma z}\hat{y} \qquad \mathbf{H}(z,t) \approx \sqrt{\frac{2\pi f\mu}{\sigma}}E_0 e^{-\alpha z}\cos\left(\omega t - \beta z + \frac{\pi}{4}\right)\hat{y}$$

7.6 Imperfect dielectric (small σ : $\dfrac{\sigma}{\omega\varepsilon} \ll 1$, $\mu = \mu_r\mu_0$, $\varepsilon = \varepsilon_r\varepsilon_0$)

Taylor Approximation will be used frequenctly here

$$\gamma = \sqrt{j\omega\mu\left(\sigma + j\omega\varepsilon\right)}$$

Remark. $\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} \approx \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1}-1\right]} = 0$ is not a good ,so use Taylor Series (at zero)

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + ...$$

Taylor Expand $\sqrt{1+x}$ for small x at $x = 0$. Since x is small, so terms larger than x^3 can be ignore (actually x^2 can also be ignored , but it is not deleted for better approximation)

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

Thus

$$\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} \approx 1+\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2-\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^4$$

$$\alpha=\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} \approx \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2-\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^4-1\right]}$$

$$=\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}\left[1-\frac{1}{4}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]}=\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\sqrt{\left[1-\frac{1}{4}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]}$$

Taylor Approximate $\sqrt{1-\frac{1}{4}\left(\frac{\sigma}{\omega\varepsilon}\right)^2}$ again (this time up to power 2 only)

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\left[1-\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]$$

$$\beta=\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} \approx \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2-\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^4+1\right]}=\omega\sqrt{\mu\varepsilon}\sqrt{1+\frac{1}{4}\left(\frac{\sigma}{\omega\varepsilon}\right)^2-\frac{1}{16}\left(\frac{\sigma}{\omega\varepsilon}\right)^4}$$

$$\beta=\omega\sqrt{\mu\varepsilon}\sqrt{1+\frac{1}{4}\left(\frac{\sigma}{\omega\varepsilon}\right)^2-\underbrace{\frac{1}{16}\left(\frac{\sigma}{\omega\varepsilon}\right)^4}_{ignore}} \approx \omega\sqrt{\mu\varepsilon}\sqrt{1+\frac{1}{4}\left(\frac{\sigma}{\omega\varepsilon}\right)^2} \overset{\text{Taylor}}{\approx} \omega\sqrt{\mu\varepsilon}\left[1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]$$

$$v_p=\frac{\omega}{\beta}\approx\frac{1}{\sqrt{\mu\varepsilon}\left[1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]}=\frac{1}{\sqrt{\mu\varepsilon}}\frac{1}{1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2}$$

Taylor aprroximation on the term $\frac{1}{1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2}$

$$\frac{1}{1+x^2}\approx\frac{1}{1+x^2}|_{x=0}+\underbrace{\left[(-1)(1+x^2)^{-2}(2x)|_{x=0}\right]}_{0}x+\frac{(-1)(1+x^2)^{-2}(2)|_{x=0}}{2!}x$$

i.e.

$$\frac{1}{1+x^2}\approx 1-x$$

So

$$\frac{1}{1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2}\approx 1-\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2$$

Thus

$$v_p \approx= \frac{1}{\sqrt{\mu\varepsilon}}\left[1-\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]$$

$$\lambda = \frac{2\pi}{\beta} \approx \frac{2\pi}{\sqrt{\mu\varepsilon}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]} \approx \frac{2\pi}{\sqrt{\mu\varepsilon}}\left[1 - \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]$$

Finally

$$\eta = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{\frac{1}{4}}} \exp\left(j\frac{1}{2}\tan^{-1}\frac{\sigma}{\varepsilon\omega}\right)$$

Remark. $\exp\left(j\frac{1}{2}\tan^{-1}\frac{\sigma}{\varepsilon\omega}\right) \approx \exp\left(j\frac{1}{2}\tan^{-1} 0\right) = e^{j0} = 1$is not good

The Taylor approximation of η is very long, for simplicity, ignoe all terms that power ≥ 3

Consider Taylor approximation on $\tan^{-1} x$

Let $y = \tan^{-1} x \iff \tan y = x \iff \sec^2 y \frac{dy}{dx} = 1 \iff \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$

And

$$\frac{d^2}{dx^2}\tan^{-1} x = \frac{d}{dx}\frac{1}{1 + x^2} = \frac{-2x}{(1 + x^2)^2} \qquad \text{It is zero when } x = 0 \text{ so no } x^2 \text{ term}$$

Thus

$$\tan^{-1} x \approx \underbrace{\tan^{-1} x|_{x=0}}_{0} + \frac{1}{1 + x^2}|_{x=0} x + \underbrace{\frac{\frac{-2x}{(1 + x^2)^2}|_{x=0}}{2!}}_{x=0} x^2$$

$$\tan^{-1} x \approx x \qquad \Longrightarrow \tan^{-1}\frac{\sigma}{\varepsilon\omega} \approx \frac{\sigma}{\varepsilon\omega}$$

(Actually it is a famous approximation that , for small angle θ , $\sin\theta \approx \tan\theta \approx \theta$)

For the exp term

$$\exp\left(j\frac{1}{2}\tan^{-1}\frac{\sigma}{\varepsilon\omega}\right) \approx \exp\left(j\frac{1}{2}\frac{\sigma}{\varepsilon\omega}\right) = \exp\left(j\frac{\sigma}{2\varepsilon\omega}\right)$$

Since $e^x \approx 1 + x + \frac{x^2}{2!}$

$$\exp\left(j\frac{1}{2}\tan^{-1}\frac{\sigma}{\varepsilon\omega}\right) \approx 1 + j\frac{\sigma}{2\varepsilon\omega} + \frac{-\left[\frac{\sigma}{2\varepsilon\omega}\right]^2}{2!} = 1 + j\frac{\sigma}{2\varepsilon\omega} - \frac{1}{8}\left[\frac{\sigma}{\varepsilon\omega}\right]^2$$

Then for $\left[1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{\frac{1}{4}}$, consider $(1 + x^2)^{\frac{1}{4}}$

$$(1 + x^2)^{\frac{1}{4}} \approx (1 + x^2)^{\frac{1}{4}}|_{x=0} + \underbrace{\frac{x}{2(1 + x^2)^{\frac{3}{2}}}|_{x=0}}_{0} x + \frac{\frac{1}{2(1 + x^2)^{\frac{7}{3}}}|_{x=0}}{2!} x^2$$

$$(1+x^2)^{\frac{1}{4}} \approx 1 + \frac{1}{4}x^2 \qquad \Longrightarrow \left[1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{\frac{1}{4}} \approx 1 + \underbrace{\frac{1}{4}\left(\frac{\sigma}{\varepsilon\omega}\right)^4}_{ignore} \approx 1$$

Finally

$$\eta = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{\frac{1}{4}}} \exp\left(j\frac{1}{2}\tan^{-1}\frac{\sigma}{\varepsilon\omega}\right) \approx \sqrt{\frac{\mu}{\varepsilon}} \cdot 1 \cdot \left[1 + j\frac{\sigma}{2\varepsilon\omega} - \frac{1}{8}\left[\frac{\sigma}{\varepsilon\omega}\right]^2\right]$$

$$\eta \approx \sqrt{\frac{\mu}{\varepsilon}}\left[1 - \frac{1}{8}\left[\frac{\sigma}{\varepsilon\omega}\right]^2 + j\frac{\sigma}{2\varepsilon\omega}\right]$$

$$|\eta| = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\left(1 - \frac{1}{8}\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right)^2 + \left(\frac{\sigma}{2\varepsilon\omega}\right)^2} \qquad \angle\eta = \tan^{-1}\frac{\frac{\sigma}{2\varepsilon\omega}}{1 - \frac{1}{8}\left(\frac{\sigma}{\varepsilon\omega}\right)^2}$$

Where are

$$\widetilde{\mathbf{E}}(z) = E_0 e^{-\gamma z}\hat{x} \qquad \mathbf{E}(z,t) \approx E_0 \exp\left[-\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\left[1 - \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]z\right]\cos\left(\omega t - \omega\sqrt{\mu\varepsilon}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]z\right)\hat{x}$$

$$\widetilde{\mathbf{H}}(z) \approx \eta E_0 e^{-\gamma z}\hat{y} \qquad \mathbf{H}(z,t) \approx |\eta| E_0 \exp\left[-\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\left[1 - \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]z\right]\cos\left(\omega t - \omega\sqrt{\mu\varepsilon}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]z + \angle\eta\right)\hat{y}$$