# Electromagnetic Wave II

Andersen Ang

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- Polarization of Plane Wave
- Loss Tangent, Skin effect, Skin depth

## 1 Polarizations of Plane Wave

The polarization is the locus of the tip of the E-field. The parameters that affect the polarization of the plane wave include : amplitudes, frequency, phase

### 1.1 E-fields with same frequency, same amplitude, same phase

A **E** field of a plane wave that is x-polarized $\mathbf{E_x}(z,t) = E_0 \sin(\omega_0 t - kz)\,\hat{x}$
A **E** field of another plane wave with same frequency that is y-polarized $\mathbf{E}_y(z,t) = E_0 \sin(\omega_0 t - kz)\,\hat{y}$
The resultant **E** field has magnitude of

$$E(z,t) = \sqrt{\mathbf{E}_x^2(z,t) + \mathbf{E}_y^2(z,t)} = E_0 \sin(\omega_0 t - kz)$$

The angle $\alpha$ between resultant vector and x-axis is

$$\alpha = \tan^{-1}\frac{|\mathbf{E_y}(z,t)|}{|\mathbf{E_x}(z,t)|} = \tan^{-1} 1 = \frac{\pi}{4}$$

- The magnitude is varying
- The angle $\alpha = \frac{\pi}{4}$ is constant and time independent, so the resultant field is also linear polarized

### 1.2 E-fields with same frequency, different amplitude, same phase

A **E** field of a plane wave that is x-polarized $\mathbf{E_x}(z,t) = E_{xm} \sin(\omega_0 t - kz)\,\hat{x}$
A **E** field of another plane wave with same frequency that is y-polarized $\mathbf{E}_y(z,t) = E_{ym} \sin(\omega_0 t - kz)\,\hat{y}$
The resultant **E** field has magnitude of

$$E(z,t) = \sqrt{\mathbf{E}_x^2(z,t) + \mathbf{E}_y^2(z,t)} = \sqrt{E_{xm}^2 + E_{ym}^2}\sin(\omega_0 t - kz)$$

The angle $\alpha$ between resultant vector and x-axis is

$$\alpha = \tan^{-1}\frac{|\mathbf{E_y}(z,t)|}{|\mathbf{E_x}(z,t)|} = \tan^{-1}\frac{E_{ym}\sin(\omega_0 t - kz)}{E_{xm}\sin(\omega_0 t - kz)} = \tan^{-1}\frac{E_{ym}}{E_{xm}}$$

- The magnitude is varying
- The angle $\alpha$ a constant ( any angle between $-\pi, +\pi$ ), is time-independent, so the resultant field is also linear polarized

### 1.3 E-fields with same frequency, same amplitude, different phase with a $\frac{\pi}{2}$ lag.

A **E** field of a plane wave that is x-polarized $\mathbf{E_x}(z,t) = E_0 \sin(\omega_0 t - kz)\,\hat{x}$
A **E** field of another plane wave with same frequency with phase difference $\frac{\pi}{2}$ in y-polarized

$$\mathbf{E}_y(z,t) = E_0 \sin\left(\omega_0 t - kz + \frac{\pi}{2}\right)\hat{y} = E_0 \cos(\omega_0 t - kz)\,\hat{y}$$

The resultant **E** has the magnitude of

$$E(z,t) = \sqrt{\mathbf{E}_x^2(z,t) + \mathbf{E}_y^2(z,t)} = E_0$$

The angle $\alpha$ between resultant vector and x-axis is

$$\alpha = \tan^{-1}\frac{|\mathbf{E_y}(z,t)|}{|\mathbf{E_x}(z,t)|} = \tan^{-1}\frac{E_0 \cos(\omega_0 t - kz)}{E_0 \sin(\omega_0 t - kz)}$$

$$= \tan^{-1}\cot(\omega_0 t - kz) = \tan^{-1}\tan\left[\frac{\pi}{2} - (\omega_0 t - kz)\right]$$

$$\alpha = \frac{\pi}{2} - (\omega_0 t - kz)$$

- Magnitude of resultant **E** field is constant
- The angle $\alpha$ is a function of time and displacement, so it is varaying in a circular way. The wave is circular polarized.

### 1.4 General case

A **E** field of a plane wave that is x-polarized $\mathbf{E_x}(z,t) = E_{xm} \sin(\omega_0 t - kz + \phi_x)\,\hat{x}$
A **E** field of a plane wave that is y-polarized $\mathbf{E_y}(z,t) = E_{ym} \sin(\omega_0 t - kz + \phi_y)\,\hat{y}$
The resultant **E** has the magnitude of

$$E(z,t) = \sqrt{\mathbf{E}_x^2(z,t) + \mathbf{E}_y^2(z,t)} = \sqrt{E_{xm}^2 \sin(\omega_0 t - kz + \phi_x)^2 + E_{ym}^2 \sin(\omega_0 t - kz + \phi_y)^2}$$

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The angle $\alpha$ between resultant vector and x-axis is

$$\alpha = \tan^{-1}\frac{|\mathbf{E_y}(z,t)|}{|\mathbf{E_x}(z,t)|} = \tan^{-1}\frac{E_{ym} \sin(\omega_0 t - kz + \phi_y)}{E_{xm} \sin(\omega_0 t - kz + \phi_x)}$$

There is no close form in time domain, phasor form should be used.

# 2 Loss tangent, skin effecy, skin depth

### 2.1 Complex Permeativity

Consider the phasor form Ampere's Law

$$\nabla \times \widetilde{\mathbf{H}} = \sigma\widetilde{\mathbf{E}} + j\omega\varepsilon\widetilde{\mathbf{E}} = (\sigma + j\omega\varepsilon)\widetilde{\mathbf{E}} = j\omega\varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)\widetilde{\mathbf{E}} = j\omega\underbrace{\varepsilon\left(1 - j\left(\frac{\sigma}{\omega\varepsilon}\right)\right)}_{\varepsilon_c}\widetilde{\mathbf{E}} = j\omega\varepsilon_c\widetilde{\mathbf{E}}$$

The complex permeativity

$$\varepsilon_c = \varepsilon\left[1 - j\left(\frac{\sigma}{\omega\varepsilon}\right)\right]$$

$$|\varepsilon_c| = \varepsilon\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} \qquad \angle\varepsilon_c = \tan^{-1}\frac{\sigma}{\omega\varepsilon}$$

$$\Re e\left[\varepsilon_c\right] = \varepsilon \qquad \Im m\left[\varepsilon_c\right] = -\frac{\sigma}{\omega}$$

$$\varepsilon_c = \varepsilon\left[1 - j\left(\frac{\sigma}{\omega\varepsilon}\right)\right] = \varepsilon + j\left(-\frac{\sigma}{\omega}\right) = \varepsilon\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2}\exp\left[j\tan^{-1}\frac{\sigma}{\omega\varepsilon}\right]$$

Current ratio - Conduction current to displacement current

$$\frac{\left|\widetilde{\mathbf{J}}_c\right|}{\left|\widetilde{\mathbf{J}}_D\right|} = \frac{\left|\sigma\widetilde{\mathbf{E}}\right|}{\left|j\omega\varepsilon\widetilde{\mathbf{E}}\right|} = \frac{|\sigma|\left|\widetilde{\mathbf{E}}\right|}{|j|\cdot|\omega\varepsilon|\left|\widetilde{\mathbf{E}}\right|} = \left|\frac{\sigma}{\omega\varepsilon}\right|$$

- When frequency increase, conduction current to displacement current ratio decrease

## 2.2 Complex Permeativity for more compact parameters

$\tan\delta = \dfrac{\sigma}{\varepsilon\omega}$

$$\varepsilon_c = \varepsilon\left[1 - j\left(\frac{\sigma}{\omega\varepsilon}\right)\right] \qquad \begin{cases} j\omega\varepsilon_c = [j\omega\varepsilon + \sigma] \\ |\varepsilon_c| = \varepsilon\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} \\ \end{cases}$$

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \sqrt{(j\omega\mu)(j\omega\varepsilon_c)} = j\omega\sqrt{\mu\varepsilon_c}$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} = \omega\sqrt{\frac{\mu}{2}\left[\varepsilon\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - \varepsilon\right]} = \omega\sqrt{\frac{\mu}{2}\left[|\varepsilon_c| - \varepsilon\right]}$$

$$= \omega\sqrt{\frac{\mu}{2}\left[|\varepsilon_c| - |\varepsilon_c|\cos\theta\right]} = \omega\sqrt{\mu|\varepsilon_c|\frac{1 - \cos\theta}{2}}$$

$\cos 2a = 2\cos^2 - 1 = 1 - 2\sin^2 a$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]} = \omega\sqrt{\frac{\mu}{2}\left[\varepsilon\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + \varepsilon\right]} = \omega\sqrt{\frac{\mu}{2}\left[|\varepsilon_c| + \varepsilon\right]}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{4\pi}{\sqrt{\mu\varepsilon\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \frac{\sqrt{\mu/\varepsilon}}{\left[1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{\frac{1}{4}}} \exp\left(j\frac{1}{2}\tan^{-1}\frac{\sigma}{\varepsilon\omega}\right)$$

## 2.3 Skin Depth and Skin effect

Consider the dacay factor $e^{-\alpha z}$. When

$$e^{-\alpha z} = e^{-1} \qquad \alpha z = 1 \qquad z = \frac{1}{\alpha} \triangleq \delta$$

For good conductor ( large $\sigma$ )

$$\alpha = \sqrt{\pi f \mu \sigma}$$

$$\text{Skin depth: } \delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

**Skin effect**: the effective resistance of the conductor to increase at higher frequencies where the skin depth is smaller, thus reducing the effective cross-section of the conductor. The electric current flows mainly at the skin of the conductor with the depth $\delta$.

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