Real part and Imaginary Part of the Propagation constant γ in Electromagnetics

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Reference

David Griffiths *Introduction to Electrodynamics*
David M. Pozar *Microwave Engineering*

1 Review of some related mathematics

1.1 Square root of $z = a + jb$

The square root of a complex number will appear in propagation constant, here is the general form :

$$\gamma = \sqrt{a+jb} = \sqrt{\frac{r+a}{2}} + j\,\mathrm{Sgn}\,(b)\sqrt{\frac{r-a}{2}}$$

where $r = \sqrt{a^2+b^2}$

Thus , the real part of the complex number γ is $\left(\sqrt{\frac{r+a}{2}}\right)$, and the imaginary part is $\left(\mathrm{Sgn}\,(b)\sqrt{\frac{r-a}{2}}\right)$

Proof. **The First Proof, using Rectangular Cooredinate**

$$\sqrt{a+jb} = x + jy \iff a + jb = x^2 - y^2 + j2xy \iff \begin{cases} a = x^2 - y^2 & (*) \\ b = 2xy & (**) \end{cases}$$

Consider $a^2 + b^2$

$$a^2 + b^2 = x^4 + y^4 - 2x^2y^2 + 4x^2y^2 = x^4 + y^4 + 2x^2y^2 = \left(x^2+y^2\right)^2$$

$$\therefore x^2 + y^2 = \pm\sqrt{a^2+b^2} = \pm r \qquad (***)$$

Since $x\,,\ y \in \mathbb{R}$

$$x^2 + y^2 = +\sqrt{a^2+b^2} = +r$$

($-\sqrt{a^2+b^2}$ is rejected)

Consider $(*) + (***)$

$$2x^2 = a + \sqrt{a^2+b^2} \qquad \Rightarrow \qquad x = \pm\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}$$

Consider $(***) - (*)$

$$2y^2 = -a + \sqrt{a^2+b^2} \qquad \Rightarrow \qquad y = \pm\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}$$

From equation $(**)$,

$$\begin{cases} x\,,\ y \text{ same sign if } b > 0 \\ x\,,\ y \text{ different sign if } b < 0 \end{cases}$$

Thus

$$\gamma = x + jy = \begin{cases} \pm\left[\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}} + j\sqrt{\dfrac{-a+\sqrt{a^2+b^2}}{2}}\right] & \text{if } b > 0 \\ \pm\left[\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}} - j\sqrt{\dfrac{-a+\sqrt{a^2+b^2}}{2}}\right] & \text{if } b < 0 \end{cases}$$

Using $r = \sqrt{a^2+b^2}$ and $\text{sgn}(b)$,

$$\gamma = \pm\left[\sqrt{\frac{r+a}{2}} + j\,\text{Sgn}\,(b)\sqrt{\frac{r-a}{2}}\right]$$

$\pm$ sign is related to the direction / determined by direction
so finally

$$\gamma = \sqrt{\frac{r+a}{2}} + j\,\text{Sgn}\,(b)\sqrt{\frac{r-a}{2}}$$

The second proof, using polar coordinate

$$z = a + jb = r\exp\left[j\tan^{-1}\frac{b}{a}\right] \qquad r = \sqrt{a^2+b^2}$$

$$\sqrt{z} = \sqrt{r}\exp\left[\frac{j}{2}\tan^{-1}\frac{b}{a}\right] = \sqrt{r}\cos\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right] + j\sqrt{r}\sin\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right]$$

The sign of b is important , since $\tan^{-1}\left(\dfrac{b}{a}\right)$ give out a angle, and $\sin(\pm\theta) = \pm\sin\theta$:

$$\sqrt{z} = \begin{cases} \sqrt{r}\cos\left[\dfrac{1}{2}\tan^{-1}\frac{b}{a}\right] + j\sqrt{r}\sin\left[\dfrac{1}{2}\tan^{-1}\frac{b}{a}\right] & b > 0 \\ \sqrt{r}\cos\left[\dfrac{1}{2}\tan^{-1}\frac{b}{a}\right] - j\sqrt{r}\sin\left[\dfrac{1}{2}\tan^{-1}\frac{b}{a}\right] & b < 0 \end{cases}$$

Using Sgn notation on b

$$\sqrt{z} = \sqrt{r}\cos\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right] + j\text{Sgn}(b)\sqrt{r}\sin\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right]$$

Turn sin into cos

$$= \sqrt{r}\cos\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right] + j\text{Sgn}(b)\sqrt{r}\sqrt{1-\cos^2\left[\frac{1}{2}\tan^{-1}\frac{b}{a}\right]}$$

Recall, the Double Angle Formula : $\cos 2A = 2\cos^2 A - 1 \iff \cos A = \sqrt{\dfrac{\cos 2A + 1}{2}}$

$$= \sqrt{r}\sqrt{\frac{\cos\left(\tan^{-1}\frac{b}{a}\right)+1}{2}} + j\mathrm{Sgn}(b)\sqrt{r}\sqrt{1-\frac{\cos\left(\tan^{-1}\frac{b}{a}\right)+1}{2}}$$

$$= \sqrt{r}\sqrt{\frac{\cos\left(\tan^{-1}\frac{b}{a}\right)+1}{2}} + j\mathrm{Sgn}(b)\sqrt{r}\sqrt{\frac{\cos\left(\tan^{-1}\frac{b}{a}\right)-1}{2}}$$

For $\tan\theta = \frac{b}{a}$, $\cos\theta = \frac{a}{\sqrt{a^2+b^2}} = \frac{a}{r}$

$$= \sqrt{r}\sqrt{\frac{\frac{a}{r}+1}{2}} + j\mathrm{Sgn}(b)\sqrt{r}\sqrt{\frac{\frac{a}{r}-1}{2}} \;=\; \sqrt{\frac{a+r}{2}} + j\mathrm{Sgn}(b)\sqrt{\frac{a-r}{2}}$$

$\therefore$

$$\gamma = \sqrt{a+jb} = \sqrt{\frac{r+a}{2}} + j\,\mathrm{Sgn}\,(b)\,\sqrt{\frac{r-a}{2}}$$

□

1.2 Curl of Curl

$$\nabla \times \nabla \times A = \nabla\left(\nabla \cdot A\right) - \nabla^2 A$$

Proof. (This is a proof by brute force, just expand everything from definition)

$$\nabla \times A = \det\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_X & A_Y & A_Z \end{pmatrix} = \left\langle \left(\frac{\partial A_Z}{\partial y} - \frac{\partial A_Y}{\partial z}\right), \left(\frac{\partial A_X}{\partial z} - \frac{\partial A_Z}{\partial x}\right), \left(\frac{\partial A_Y}{\partial x} - \frac{\partial A_X}{\partial y}\right) \right\rangle$$

$$\nabla \times \nabla \times A = \det\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\nabla \times A)_X & (\nabla \times A)_Y & (\nabla \times A)_Z \end{pmatrix}$$

$$= \left\langle \left(\frac{\partial\left(\nabla \times A\right)_Z}{\partial y} - \frac{\partial\left(\nabla \times A\right)_Y}{\partial z}\right), \left(\frac{\partial\left(\nabla \times A\right)_X}{\partial z} - \frac{\partial\left(\nabla \times A\right)_Z}{\partial x}\right), \left(\frac{\partial\left(\nabla \times A\right)_Y}{\partial x} - \frac{\partial\left(\nabla \times A\right)_X}{\partial y}\right) \right\rangle$$

$$= \left\langle \left(\frac{\partial^2 A_Y}{\partial y \partial x} - \frac{\partial^2 A_X}{\partial y^2} + \frac{\partial^2 A_Z}{\partial x \partial z} - \frac{\partial^2 A_X}{\partial z^2}\right), \left(\frac{\partial^2 A_Z}{\partial z \partial y} - \frac{\partial^2 A_Y}{\partial z^2} - \frac{\partial^2 A_Y}{\partial x^2} + \frac{\partial^2 A_X}{\partial x \partial y}\right), \left(\frac{\partial^2 A_X}{\partial x \partial z} - \frac{\partial^2 A_Z}{\partial x^2} - \frac{\partial^2 A_Z}{\partial y^2} + \frac{\partial^2 A_Y}{\partial y \partial z}\right) \right\rangle$$

$$\nabla \cdot A = \frac{\partial A_X}{\partial x} + \frac{\partial A_Y}{\partial y} + \frac{\partial A_Z}{\partial z}$$

$$\nabla\left(\nabla \cdot A\right) = \left\langle \frac{\partial}{\partial x}\left(\frac{\partial A_X}{\partial x} + \frac{\partial A_Y}{\partial y} + \frac{\partial A_Z}{\partial z}\right), \frac{\partial}{\partial y}\left(\frac{\partial A_X}{\partial x} + \frac{\partial A_Y}{\partial y} + \frac{\partial A_Z}{\partial z}\right), \frac{\partial}{\partial z}\left(\frac{\partial A_X}{\partial x} + \frac{\partial A_Y}{\partial y} + \frac{\partial A_Z}{\partial z}\right) \right\rangle$$

$$\nabla(\nabla\cdot A)=\left\langle\left(\frac{\partial^2A_X}{\partial x^2}+\frac{\partial^2A_Y}{\partial x\partial y}+\frac{\partial^2A_Z}{\partial x\partial z}\right),\left(\frac{\partial^2A_X}{\partial y\partial x}+\frac{\partial^2A_Y}{\partial y^2}+\frac{\partial^2A_Z}{\partial x\partial z}\right),\left(\frac{\partial^2A_X}{\partial z\partial x}+\frac{\partial^2A_Y}{\partial z\partial y}+\frac{\partial^2A_Z}{\partial z^2}\right)\right\rangle$$

$$\nabla^2A=\left\langle\left(\nabla^2A_X\right),\left(\nabla^2A_Y\right),\left(\nabla^2A_Z\right)\right\rangle$$

$$=\left\langle\left(\frac{\partial^2A_X}{\partial x^2}+\frac{\partial^2A_X}{\partial y^2}+\frac{\partial^2A_X}{\partial z^2}\right),\left(\frac{\partial^2A_Y}{\partial x^2}+\frac{\partial^2A_Y}{\partial y^2}+\frac{\partial^2A_Y}{\partial z^2}\right),\left(\frac{\partial^2A_Z}{\partial x^2}+\frac{\partial^2A_Z}{\partial y^2}+\frac{\partial^2A_Z}{\partial z^2}\right)\right\rangle$$

$\therefore\quad \nabla(\nabla\cdot A)-\nabla^2A=$

$$\left\langle\left(\frac{\partial^2A_Y}{\partial x\partial y}+\frac{\partial^2A_Z}{\partial x\partial z}-\frac{\partial^2A_X}{\partial y^2}-\frac{\partial^2A_X}{\partial z^2}\right),\left(\frac{\partial^2A_X}{\partial y\partial x}+\frac{\partial^2A_Z}{\partial x\partial z}-\frac{\partial^2A_Y}{\partial x^2}-\frac{\partial^2A_Y}{\partial z^2}\right),\left(\frac{\partial^2A_X}{\partial z\partial x}+\frac{\partial^2A_Y}{\partial z\partial y}-\frac{\partial^2A_Z}{\partial x^2}-\frac{\partial^2A_Z}{\partial y^2}\right)\right\rangle$$

They are equal, thus

$$\nabla\times\nabla\times A=\nabla(\nabla\cdot A)-\nabla^2A$$

□

2 The propagation constant in EM wave

2.1 The Wave Equation

The Phasor form Maxwell's Eqautions in source free region

$$\begin{cases}\nabla\times E=-\mu\dfrac{\partial H}{\partial t} & \text{Faraday's Law}\\ \nabla\times H=\sigma E+\epsilon\dfrac{\partial E}{\partial t} & \text{Ampère's circuital law}\end{cases}$$

The vector identity : *curl of curl*

$$\nabla\times\nabla\times A=\nabla(\nabla\cdot A)-\nabla^2A$$

Take the *curl* of both equation

$$\begin{cases}\nabla(\nabla\cdot E)-\nabla^2E=-\mu\left(\sigma E+\epsilon\dfrac{\partial E}{\partial t}\right)\\ \nabla(\nabla\cdot H)-\nabla^2H=\sigma\left(-\mu\dfrac{\partial H}{\partial t}\right)+\epsilon\dfrac{\partial}{\partial t}\left(-\mu\dfrac{\partial H}{\partial t}\right)\end{cases}$$

With the help of Gauss's Law (No source)

$$\nabla\cdot E=\nabla\cdot H=0$$

The 2 Maxwell's Equation is now then

$$\begin{cases}\nabla^2E=\mu\sigma\dfrac{\partial E}{\partial t}+\mu\epsilon\dfrac{\partial^2E}{\partial t^2}\\ \nabla^2H=\mu\sigma\dfrac{\partial H}{\partial t}+\mu\epsilon\dfrac{\partial^2H}{\partial t^2}\end{cases}\qquad \textit{Helmholtz Equation}$$

Use *phasor*

$$\begin{cases} \nabla^2 E = +j\omega\mu\sigma E - \omega^2\mu\epsilon E = j\omega\mu\left(\sigma + j\omega\epsilon\right) E \\ \nabla^2 H = +j\omega\mu\sigma H - \omega^2\mu\epsilon H = j\omega\mu\left(\sigma + j\omega\epsilon\right) H \end{cases}$$

Let $\gamma = \sqrt{j\omega\mu\left(\sigma + j\omega\epsilon\right)}$

$$\begin{cases} \nabla^2 E - \gamma^2 E = 0 \\ \nabla^2 H - \gamma^2 H = 0 \end{cases} \qquad \textit{Phasor Helmholtz Equations}$$

2.2 The propagation constant

$$\gamma = \sqrt{j\omega\mu\left(\sigma + j\omega\epsilon\right)} = \alpha + j\beta$$

Recall,

$$\gamma = \sqrt{a + jb} = \sqrt{\frac{r+a}{2}} + j\text{sgn}\left(b\right)\sqrt{\frac{r-a}{2}}$$

Now

$$\gamma = \sqrt{j\omega\mu\left(\sigma + j\omega\epsilon\right)} = \sqrt{-\omega^2\mu\epsilon + j\sigma\omega\mu}$$

$$\begin{cases} a = -\omega^2\mu\epsilon \\ b = \sigma\omega\mu \in \mathbb{R}^+ \;\Rightarrow \text{sgn}\left(b\right) = +1 \\ r = \sqrt{a^2 + b^2} = \omega\mu\sqrt{\sigma^2 + \omega^2\epsilon^2} = \omega^2\mu\epsilon\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} \\ r + a = \omega^2\mu\epsilon\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - \omega^2\mu\epsilon = \omega^2\mu\epsilon\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right] \\ r - a = \omega^2\mu\epsilon\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + \omega^2\mu\epsilon = \omega^2\mu\epsilon\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right] \end{cases} \Rightarrow \begin{cases} \alpha = Re\left(\gamma\right) = \sqrt{\frac{r+a}{2}} \\ \beta = Im\left(\gamma\right) = \text{sgn}\left(b\right)\sqrt{\frac{r-a}{2}} \end{cases}$$

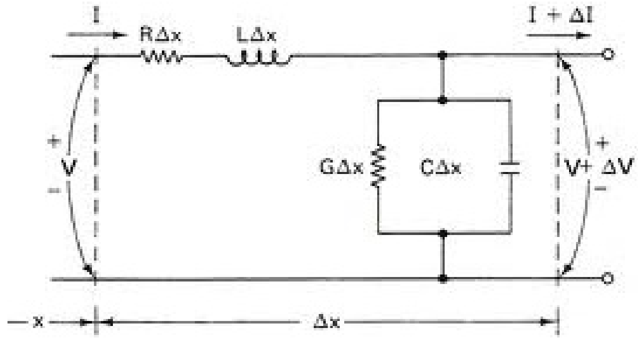
i.e.

$$\gamma = \sqrt{j\omega\mu\left(\sigma + j\omega\epsilon\right)} = \alpha + j\beta \qquad \begin{aligned} \alpha &= \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]} \\ \beta &= \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]} \end{aligned}$$

3 The propagation constant in Transmission Line Model

3.1 The Telegrapher Equation & Wave Equation from KCL, KVL

The Transmission Line model



KVL :

$$v(z,t)-R\Delta z\cdot i\left(z,t\right)-L\Delta z\cdot\frac{\partial i\left(z,t\right)}{\partial t}-v\left(z+\Delta z,t\right)=0 \quad\Rightarrow\quad \frac{v\left(z+\Delta z,t\right)-v(z,t)}{\Delta z}=-R\cdot i\left(z,t\right)-L\frac{\partial i\left(z,t\right)}{\partial t}$$

KCL : $i(z,t)-G\Delta z\cdot v\left(z+\Delta z,t\right)-C\Delta z\cdot\frac{\partial v\left(z+\Delta z,t\right)}{\partial t}-i\left(z+\Delta z,t\right)=0$

$$\Rightarrow\quad \frac{i\left(z+\Delta z,t\right)-i\left(z,t\right)}{\Delta z}=-Gv\left(z,t\right)-C\frac{\partial v\left(z+\Delta z,t\right)}{\partial t}$$

Shrink segment $\Delta z\to 0$

$$\begin{cases}\lim\limits_{\Delta z\to 0}\dfrac{v\left(z+\Delta z,t\right)-v(z,t)}{\Delta z}=\dfrac{\partial v(z,t)}{\partial z}=-Ri\left(z,t\right)-L\dfrac{\partial i\left(z,t\right)}{\partial t}\\[2ex] \lim\limits_{\Delta z\to 0}\dfrac{i\left(z+\Delta z,t\right)-i\left(z,t\right)}{\Delta z}=-Gv\left(z,t\right)-C\dfrac{\partial v\left(z,t\right)}{\partial t}\end{cases}$$

Using phasor

$$\begin{cases}\dfrac{dV(z)}{dz}=-\left(R+j\omega L\right)I(z)\\[2ex] \dfrac{dI(z)}{dz}=-\left(G+j\omega C\right)V(z)\end{cases}\qquad Telegrapher\ Equation$$

$\frac{d}{dz}\left(Telegrapher\ Equation\right)$:

$$\begin{cases}\dfrac{d^2V(z)}{dz^2}=-\left(R+j\omega L\right)\dfrac{dI(z)}{dz}\\[2ex] \dfrac{d^2I(z)}{dz^2}=-\left(G+j\omega C\right)\dfrac{dV(z)}{dz}\end{cases}\rightarrow\begin{cases}\dfrac{d^2V(z)}{dz^2}=\left(R+j\omega L\right)\left(G+j\omega C\right)V(z)\\[2ex] \dfrac{d^2I(z)}{dz^2}=\left(G+j\omega C\right)\left(R+j\omega L\right)I(z)\end{cases}\rightarrow\begin{cases}\dfrac{d^2V(z)}{dz^2}-\gamma^2V(z)=0\\[2ex] \dfrac{d^2I(z)}{dz^2}-\gamma^2I(z)=0\\ \gamma=\sqrt{\left(R+j\omega L\right)\left(G+j\omega C\right)}\end{cases}$$

3.2 The propagation constant γ

$$\gamma=\sqrt{\left(R+j\omega L\right)\left(G+j\omega C\right)}=\alpha+j\beta$$

Recall again,

$$\gamma=\sqrt{a+jb}=\sqrt{\frac{r+a}{2}}+j\mathrm{sgn}\left(b\right)\sqrt{\frac{r-a}{2}}$$

Now

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(RG - \omega^2 LC) + j(\omega LG + \omega RC)}$$

$$a = RG - \omega^2 LC \qquad b = \omega LG + \omega RC \in \mathbb{R}^+ \ \Rightarrow \operatorname{sgn}(b) = +1$$

$$\therefore r = \sqrt{a^2 + b^2} = \sqrt{(RG - \omega^2 LC)^2 + (\omega LG + \omega RC)^2} = \sqrt{R^2G^2 + \omega^4 L^2 C^2 + \omega^2 L^2 G^2 + \omega^2 R^2 C^2}$$

$$= \sqrt{G^2(R^2 + \omega^2 L^2) + C^2\omega^2(\omega^2 L^2 + R^2)} = \sqrt{(G^2 + C^2\omega^2)(R^2 + \omega^2 L^2)}$$

$$\begin{cases} r + a = \sqrt{(G^2 + C^2\omega^2)(R^2 + \omega^2 L^2)} + RG - \omega^2 LC \\ r - a = \sqrt{(G^2 + C^2\omega^2)(R^2 + \omega^2 L^2)} - RG + \omega^2 LC \end{cases} \Rightarrow \begin{cases} \alpha = Re(\gamma) = \sqrt{\dfrac{r+a}{2}} \\ \beta = Im(\gamma) = \operatorname{sgn}(b)\sqrt{\dfrac{r-a}{2}} \end{cases}$$

$\therefore$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta \qquad \begin{aligned} \alpha &= \sqrt{\frac{RG - \omega^2 LC + \sqrt{(G^2 + C^2\omega^2)(R^2 + \omega^2 L^2)}}{2}} \\ \beta &= \sqrt{\frac{-RG + \omega^2 LC + \sqrt{(G^2 + C^2\omega^2)(R^2 + \omega^2 L^2)}}{2}} \end{aligned}$$

$$\overset{-\qquad -}{END}$$