Maximum Likelihood Estimation - Amplitude of sinusoid

Andersen Ang

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1 The estimation problem

Consider a sinusoid signal in the addictive white Gaussian noise environment over N samples

$$x[n] = A\cos(2\pi f_0 n + \phi) + w[n] \;\; n = 0, 1, ..., N-1$$

- A is the amplitude, f_0 is the frequency and ϕ is the phase .
- Now assume f_0 and ϕ are known, and A is unknown, which is the parameters to be estimated : $\theta = A$. So in this case $p(x;\theta) = p(x;A)$
- w[n] is a addictive white Gaussian noise $w[n] \sim \mathcal{N}(0, \sigma^2)$.

2 The estimation for N observation

We have N observations

$$x[n] = A\cos(2\pi f_0 n + \phi) + w[n] \;\; n = 0, 1, ..., N-1$$

Each data point will have a PDF

$$p(x[k]; A,) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2}\Big(x[k] - A\cos(2\pi f_0 k + \phi)\Big)^2 \right\}$$

By the multiplication law of probability, we have

$$p(x[0], x[1], ..., x[N-1]\,;\, A) = p(x[0]; A)\, p(x[1]; A) \,...p(x[N-1]; A)$$

Denote $p(x[0], x[1], ..., x[N-1]\,;\, A)$ as $p(x; A)$, using compact notation

$$p(x; A) = \frac{1}{(2\pi\sigma^2)^{N/2}} \prod_{n=0}^{N-1} \exp\left\{ -\frac{1}{2\sigma^2}\Big(x[n] - A\cos(2\pi f_0 n + \phi)\Big)^2 \right\}$$

Since $\exp\alpha \exp\beta = \exp(\alpha+\beta)$, the likelihood function is

$$p(x; A) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{ -\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\Big(x[n] - A\cos(2\pi f_0 n + \phi)\Big)^2 \right\}$$

The log-likelihood function will be

$$\ln p(x; A) = -\frac{N}{2}\ln\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\Big(x(n) - A\cos(2\pi f_0 n + \phi)\Big)^2$$

Then

$$\frac{\partial \ln p\left(x;A\right)}{\partial A} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\Big(x\left(n\right) - A\cos\left(2\pi f_0 n + \phi\right)\Big)\cos\left(2\pi f_0 n + \phi\right)$$

$$\frac{\partial^2 \ln p\left(x;A\right)}{\partial A^2} = \frac{-1}{\sigma^2}\sum_{n=0}^{N-1}\cos^2\left(2\pi f_0 n + \phi\right)$$

The Fisher Information Matrix is

$$I\left(\theta\right) = -\mathbb{E}\left[\frac{\partial^2 \ln p\left(x;A\right)}{\partial A^2}\right] = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\cos^2\left(2\pi f_0 n + \phi\right)$$

Thus the variance of the estimrator will be

$$Var\left(\hat{A}\right) = \frac{1}{I\left(\theta\right)} = \frac{\sigma^2}{\sum_{n=0}^{N-1}\cos^2\left(2\pi f_0 n + \phi\right)}$$

Which is the variance of the addictive noise. Notice that this is exactly the Cramer-Rao Lower Bound of this estimator, that means

The lower bound of the variance of the estimator $\hat{A} = \sigma^2$