1 The estimation problem

Consider a sinusoid signal in the additive white Gaussian noise environment over \( N \) samples

\[ x[n] = A \cos (2\pi f_0 n + \phi) + w[n] \quad n = 0, 1, ..., N - 1 \]

- \( A \) is the amplitude, \( f_0 \) is the frequency and \( \phi \) is the phase.
- Now assume \( f_0 \) and \( \phi \) are known, and \( A \) is unknown, which is the parameters to be estimated: \( \theta = A \). So in this case \( p(x; \theta) = p(x; A) \)
- \( w[n] \) is an additive white Gaussian noise \( w[n] \sim N(0, \sigma^2) \).

2 The estimation for \( N \) observation

We have \( N \) observations

\[ x[n] = A \cos (2\pi f_0 n + \phi) + w[n] \quad n = 0, 1, ..., N - 1 \]

Each data point will have a PDF

\[ p(x[k]; A) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{1}{2\sigma^2} \left( x[k] - A \cos (2\pi f_0 k + \phi) \right)^2 \right\} \]

By the multiplication law of probability, we have

\[ p(x[0], x[1], ..., x[N-1]; A) = p(x[0]; A) p(x[1]; A) ... p(x[N-1]; A) \]

Denote \( p(x[0], x[1], ..., x[N-1]; A) \) as \( p(x; A) \), using compact notation

\[ p(x; A) = \frac{1}{(2\pi\sigma^2)^{N/2}} \prod_{n=0}^{N-1} \exp \left\{ -\frac{1}{2\sigma^2} \left( x[n] - A \cos (2\pi f_0 n + \phi) \right)^2 \right\} \]

Since \( \exp \alpha \exp \beta = \exp(\alpha + \beta) \), the likelihood function is

\[ p(x; A) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} \left( x[n] - A \cos (2\pi f_0 n + \phi) \right)^2 \right\} \]

The log-likelihood function will be

\[ \ln p(x; A) = -\frac{N}{2} \ln (2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{n=0}^{N-1} \left( x[n] - A \cos (2\pi f_0 n + \phi) \right)^2 \]

Then
\[ \frac{\partial \ln p (x; A)}{\partial A} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left( x(n) - A \cos (2\pi f_0 n + \phi) \right) \cos (2\pi f_0 n + \phi) \]

\[ \frac{\partial^2 \ln p (x; A)}{\partial A^2} = -\frac{1}{\sigma^2} \sum_{n=0}^{N-1} \cos^2 (2\pi f_0 n + \phi) \]

The Fisher Information Matrix is

\[ I (\theta) = -\mathbb{E} \left[ \frac{\partial^2 \ln p (x; A)}{\partial A^2} \right] = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \cos^2 (2\pi f_0 n + \phi) \]

Thus the variance of the estimator will be

\[ \text{Var} \left( \hat{A} \right) = \frac{1}{I (\theta)} = \frac{\sigma^2}{\sum_{n=0}^{N-1} \cos^2 (2\pi f_0 n + \phi)} \]

Which is the variance of the additive noise. Notice that this is exactly the Cramer-Rao Lower Bound of this estimator, that means

The lower bound of the variance of the estimator \( \hat{A} = \sigma^2 \)