ELEC3227/4247 Mid-term Quiz2 Solution with explanation

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This document explain the solution of linear block code, cyclic code, BCH code and RS code. It is suggest to read this document after revision.

Q1. Linear Code

C={000000,010101,101010,111111}
n = length of code = 6
k = length of message = 2

Explanation : the number of possible message is related to the length of message. Since the message here is binary, so

$$\text{the number of possible message} = 2^k$$

Now number of possible message = 4, so $k = 2$

Now we have a (6,2) linear block code, so the generator matrix G has the following form

$$G = [\, P_{6-2\times 2} \mid I_{2\times 2}] \qquad P = [\, 1\ 0\ 1\ 0;\ 0\ 1\ 0\ 1\,]$$

And since $GH^T = 0$, thus H^T has the following form.

$$H^T = [\, I \mid -P^T]$$

And in this case $GH^T = [\, P_{4\times2} \mid I_{2\times2}]\, [I_{4\times4} \mid -P_{2\times4}^T\,] = P - P = 0$

Because it is binary, so subtraction is same as addition, thus $H^T = [\, I \mid P^T]$

After obtaining the matrix G and H , construct the syndrome as $s = rH^T$

$$\begin{aligned} s &= rH^T \\ &= (c+e)H^T \\ &= cH^T + eH^T \\ &= mGH^T + eH^T \\ &= eH^T \end{aligned}$$

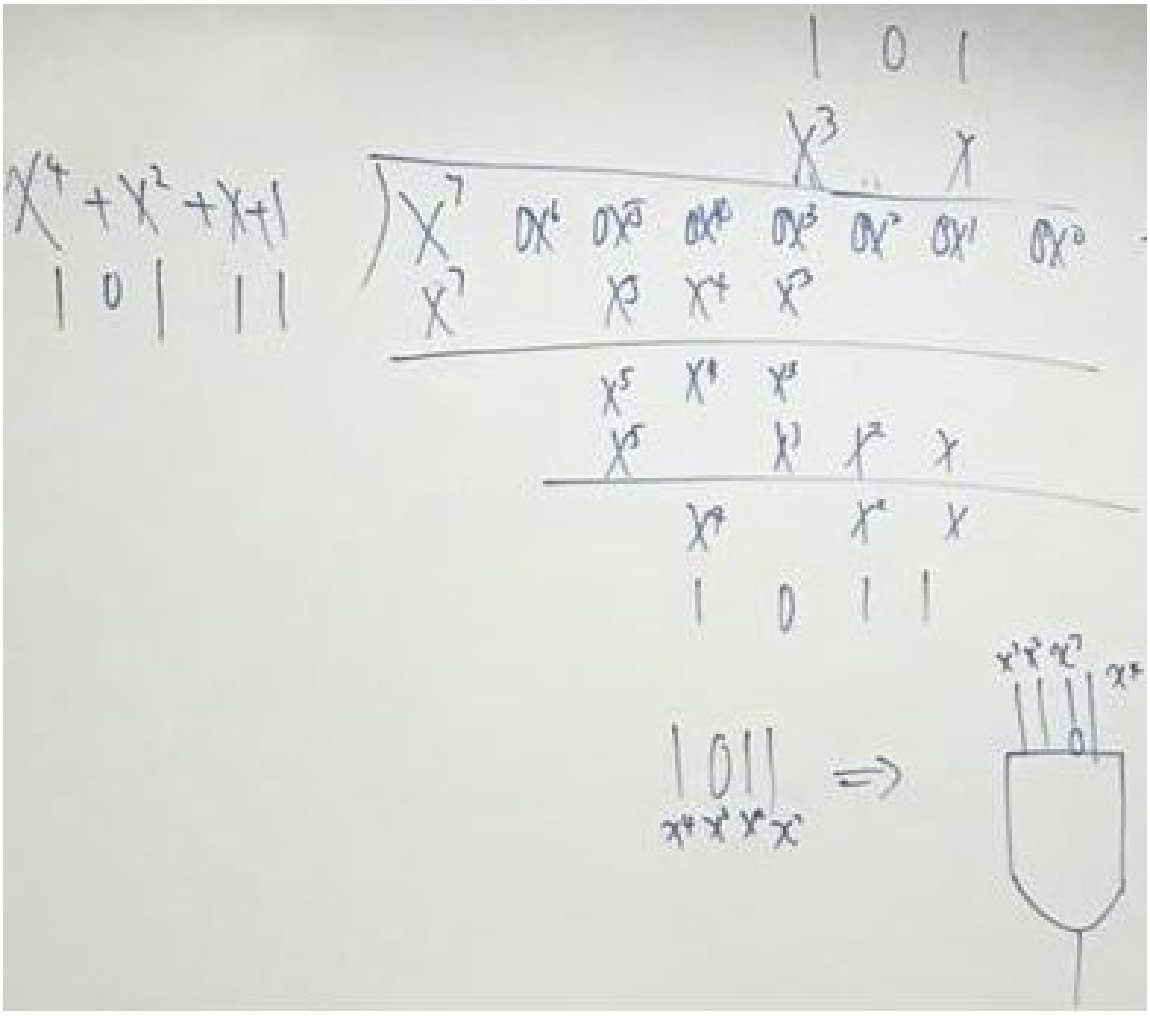
Now since e is a vector with only only one "1s". So

$$s = eH^T = [00...1...00]H^T = i-th\ row\ of\ H$$

And by comparing the s and rows of H we can find where is the error bits.

Q2.Cyclic Code

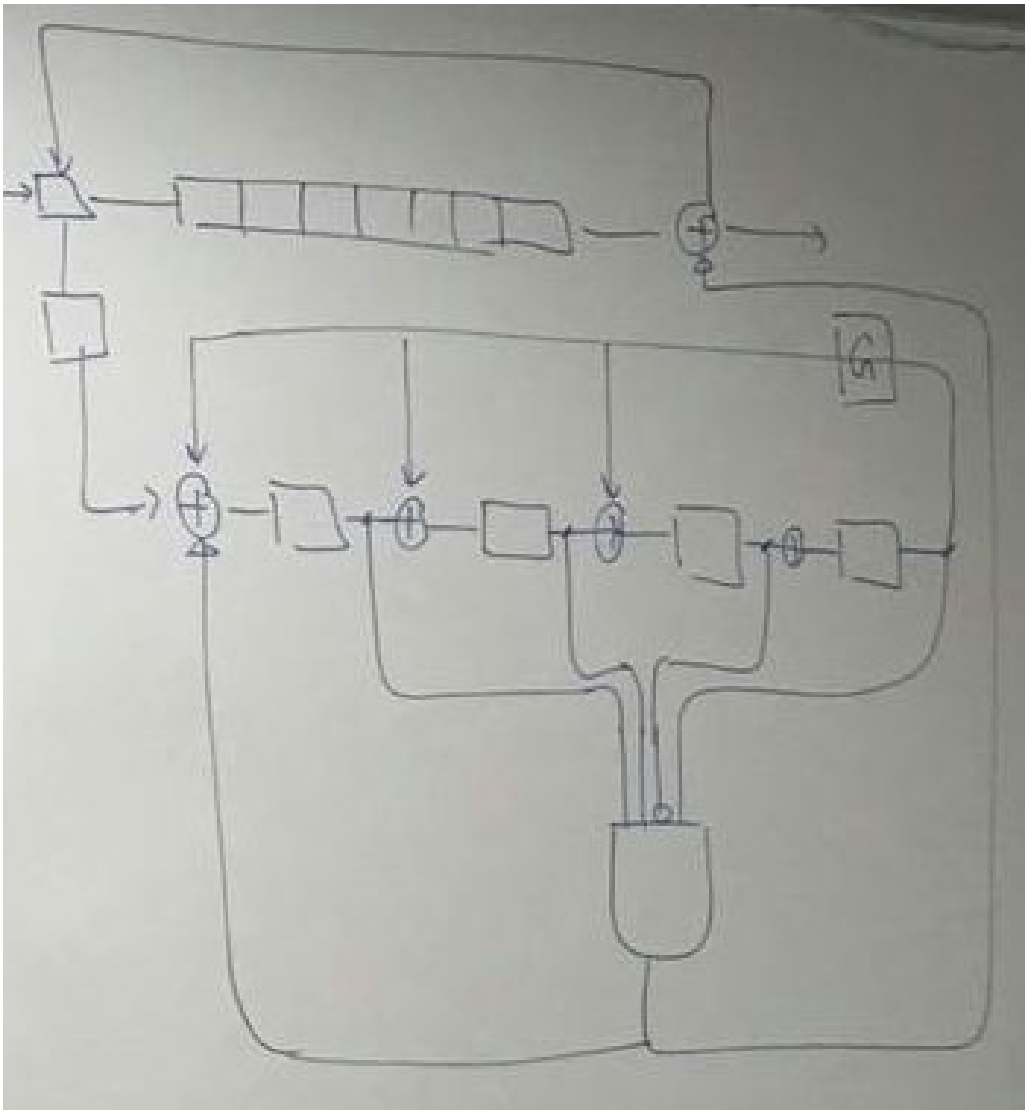
Step. 1. Compute the error detector logic gate pattern, which is equal to the remainder of $x^n \div g(x)$. The answer here is 1011, which correspond to $x^4x^3x^2x^1$. The error detector logic pattern thus is [TFTT], make sure the order of the AND gate match the expression !



Step 2. Draw the decoder.

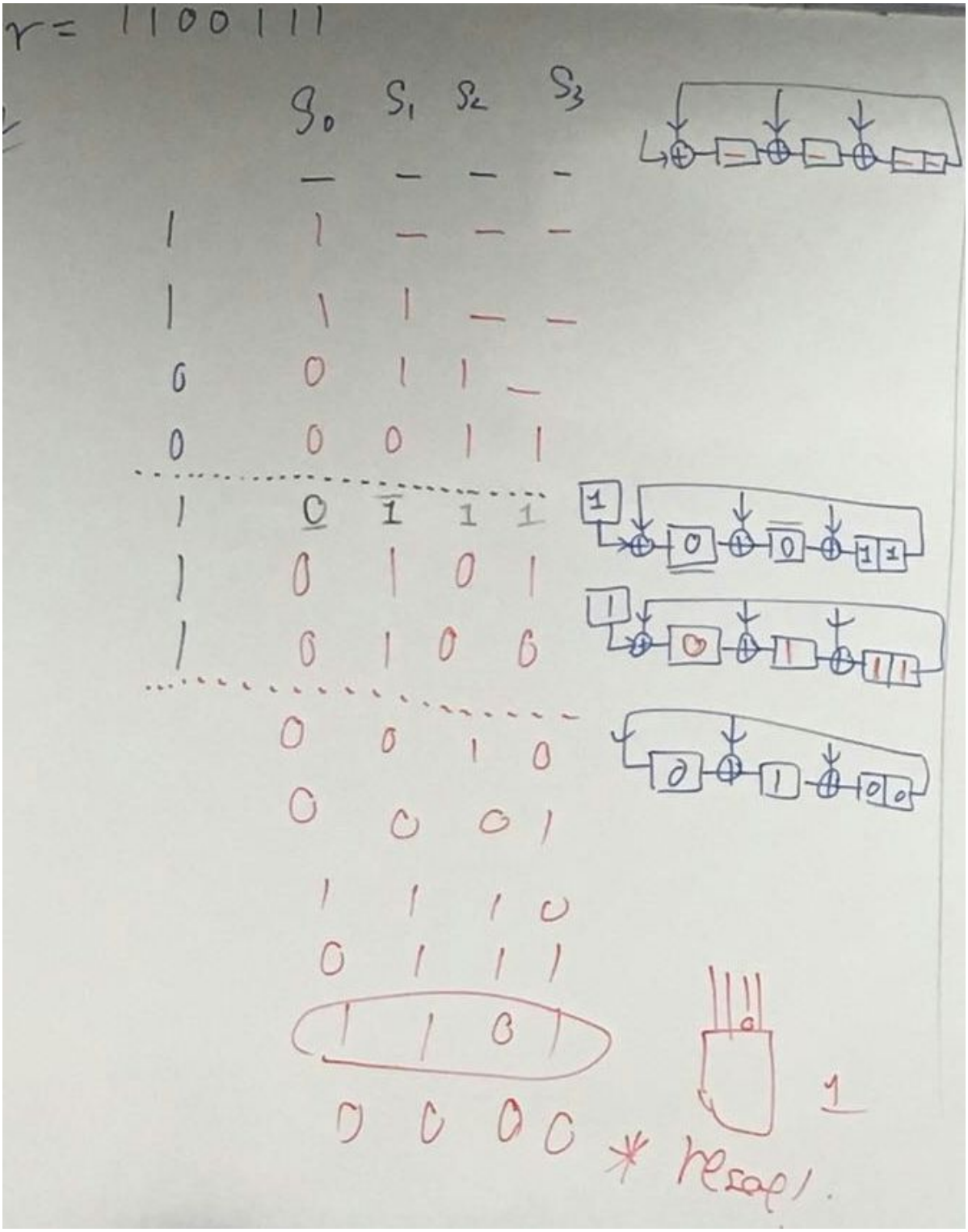
In general, the decoder consists of three parts :

1) The shift register (the uppermost part)
2) The division circuit (the middle part). The connection to the register of the division circuit is determined by the coefficient of $g(x)$
3) The error detector AND gate, the AND gate input logic pattern should match the step 1 solution.



Step 3. Compute syndrome
The received codeword *r* is fired into the whole circuit bit-by-bit and the following figure show the computation of the process.

In general there are three phase in the process , the first phase is the process of filling the unfilled register. The second phase is the phase when there is external input (from *r*) and internal input (from the feedback of the last register). The last phase is when there is no external input but only internal.



The error is detected when logic of the register match the error detector. At that moment the AND gate will fire a ‘1’ to correct the content in the shift register.

Q3. BCH Code

Step 1. Obtain the receiver polynomial $r(x)$, where the coefficients are determined by the received codeword. In here, $r(x) = x^2 + x^5 + x^{14}$

Step 2. Compute the syndrome as $S_i = r(\alpha^i)$.

i). Comput S_1, S_2,... S_{2t} according to the number of detectable error t.

If $t = 1$, it means the code can detect at most 1 error. And in this case we only need to compute S_1.

If $t = 2$, it means the code can detect at most 2 errors. And in this case we need to compute S_1 S_2 S_3 S_4.

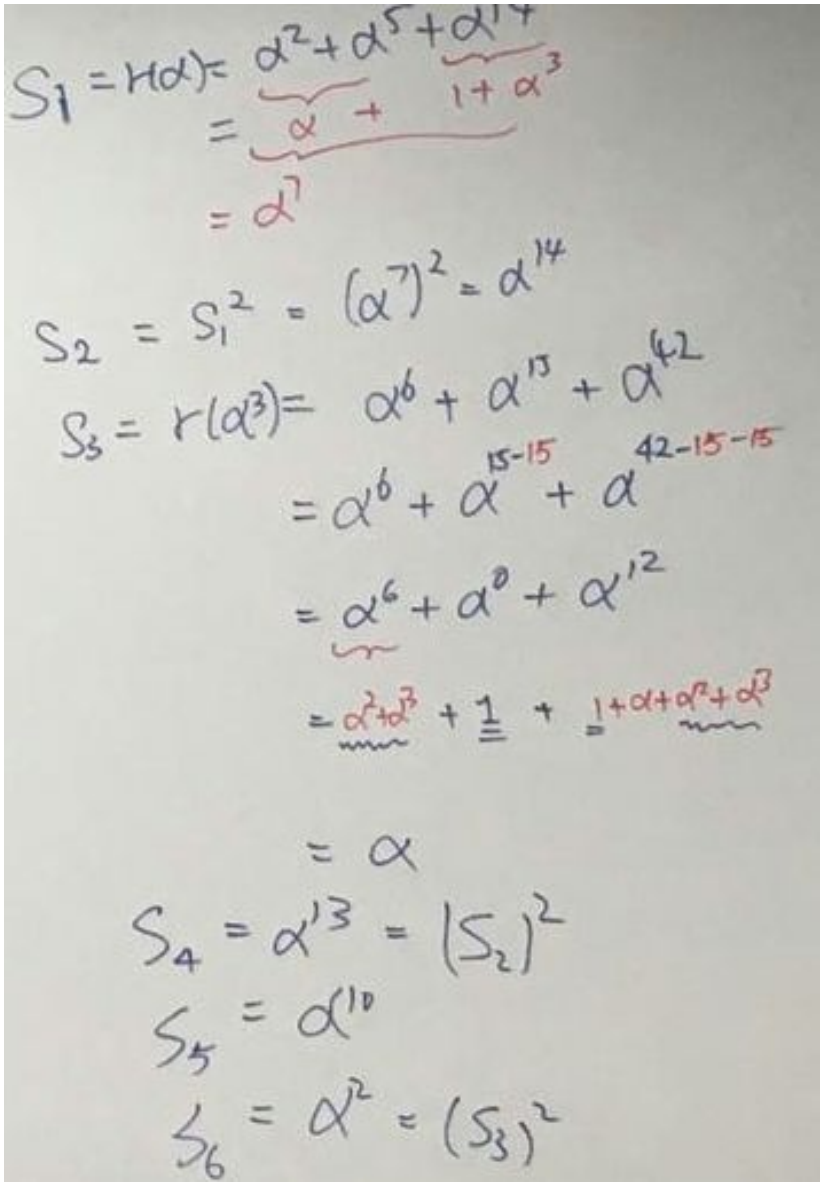
If $t = 3$, it means the code can detect at most 2 errors. And in this case we need to compute S_1 S_2 S_3 S_4 S_5 S_6.

In this question $t = 3$ so we need to determine S_1 up to S_6

ii) Compute $S_i = r(\alpha^i)$ with the helps of the table for simplification.

iii) When the power (index) of α exceed the range [*0*, *n*], add or subtract *n* until the index fall within the range [*0*, *n*].

iv) There is a short cut for BCH code , that is $S_{2k} = (S_k)^2$



Step 3. Compute the error location polynomial as $\sigma(x)$

Error location polynomial has the following form

$$\sigma(x) = 1 + \sigma_1 x + \sigma_2 x^2 + \ldots + \sigma_t x^t$$

It is a t-order polynomial, and thus $\sigma(x) = 0$ should have t roots. Recall that t is the number of detectable error (detectability), so the order of the error location polynomial is related to such detectability of the code.

The coefficients of the error location polynomial is obtained by solving Newton's identities by Peterson's method and it has different form for different t :

When $t = 1$, it can detect at most 1 error

$\sigma_1 = S_1$

$\sigma(x) = 1 + \sigma_1 x$

When $t = 2$, it can detect at most 2 errors

$\sigma_1 = S_1$

$\sigma_2 = S_1^2 + \frac{S_3}{S_1}$

$\sigma(x) = 1 + \sigma_1 x + \sigma_2 x^2$

When $t = 3$, it can detect at most 3 errors

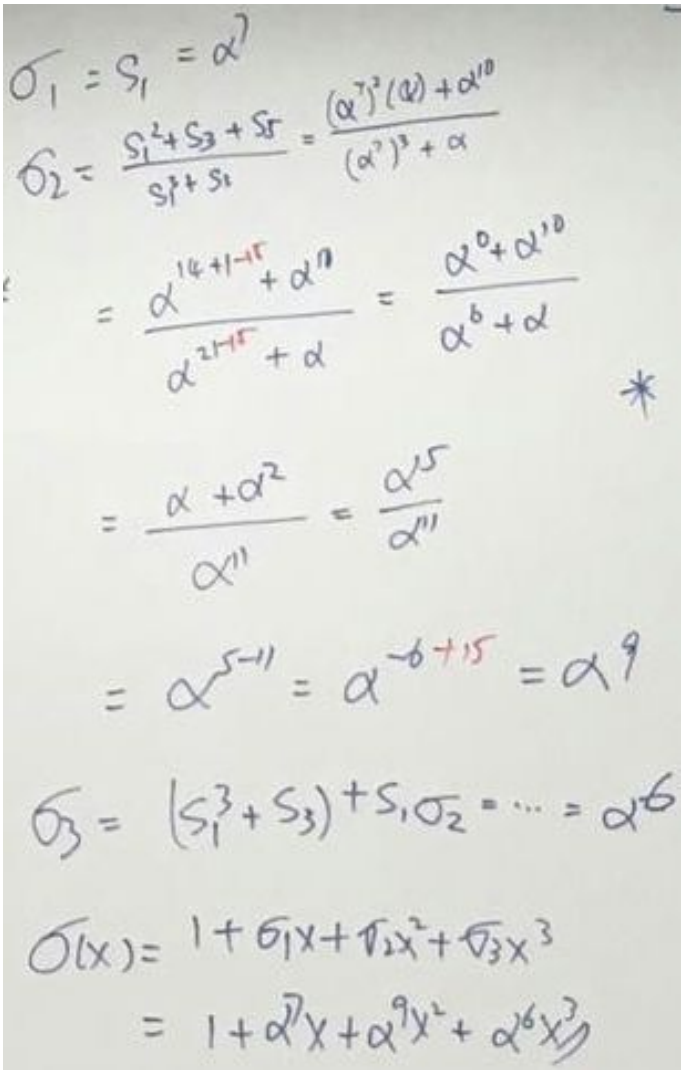
$\sigma_1 = S_1$

$\sigma_2 = \frac{S_1^2 + S_5}{S_1^3 + S_3}$

$\sigma_3 = S_1^3 + S_3 + S_1 \sigma_2$

$\sigma(x) = 1 + \sigma_1 x + \sigma_2 x^2 + \sigma_3 x^3$

In this problem, $t = 3$, so $\sigma(x) = 1 + \sigma_1 x + \sigma_2 x^2 + \sigma_3 x^3$, and the error location polynomial for this problem is



Step 4. Find the error position and correct the error

We have a *n*-bit codeword, but we count the monomial order from 0, 1, 2, … up to *n-1*, so

$$k^{th}\ bit\ is\ error \Leftrightarrow \sigma(\alpha^{n-(k-1)}) = 0$$

After checking all the $\sigma(\alpha^i)$:

$$\sigma(\alpha^1) = 0 \Leftrightarrow 15^{th}\ bit\ is\ error$$
$$\sigma(\alpha^{10}) = 0 \Leftrightarrow 6^{th}\ bit\ is\ error$$
$$\sigma(\alpha^{13}) = 0 \Leftrightarrow 3^{th}\ bit\ is\ error$$

Step 5. Chien's Searching Circuit

The Chien's Searching circuit is the hardware implementation of step 4. Since we need to compute multiplication in step 4, so Chien's searching circuit is basically a multiplication circuit, with the coefficient equal to the coefficients in the error location polynomial $\sigma(x)$.

Since t=3 in this problem, so $\sigma(x) = 1 + \sigma_1 x + \sigma_2 x^2 + \sigma_3 x^3$, and we have the register content of equal to the coefficients $\sigma_1, \sigma_2, \sigma_3$

i) Write down the register content
Refer to the table obtained from the Galois Field

$\sigma_1 = \alpha^7 = 1101$, so the content of the 4 register are

1	1	0	1

$\sigma_2 = \alpha^9 = 0101$, so the content of the 4 register are

0	1	0	1

$\sigma_3 = \alpha^6 = 0011$, so the content of the 4 register are

0	0	1	1

ii) Find the connection pattern between the register
The connection pattern between the register is determined by the expression

$$\alpha^i \beta = \alpha^i (b_0 + b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3)$$
$$= (b_0\alpha^i + b_1\alpha^{1+i} + b_2\alpha^{2+i} + b_3\alpha^{3+i})$$

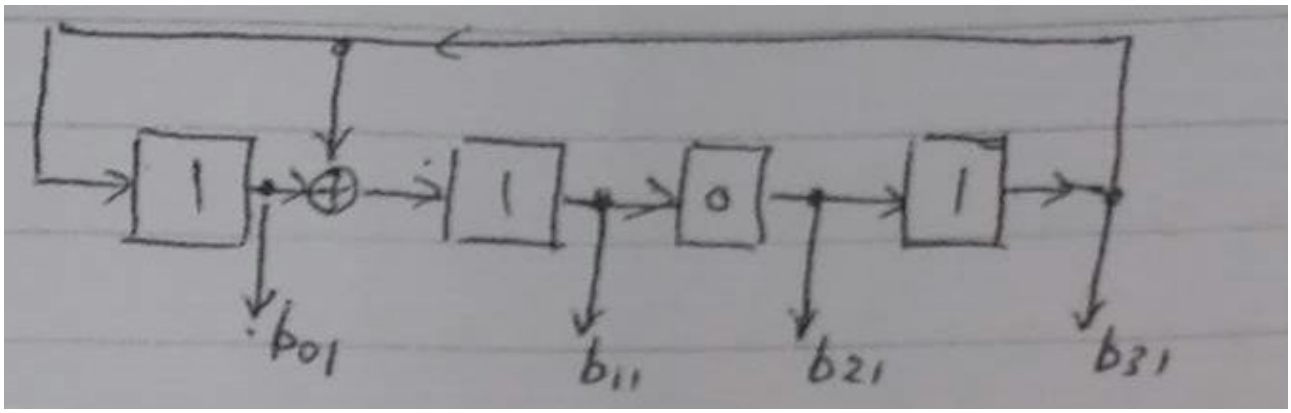
Where coefficient of b_i is the connection pattern to the register.

Now in-order to simplify $\alpha^i, \alpha^{1+i}, \alpha^{2+i}, \alpha^{3+i}$, we need to apply the minimal polynomial $\Phi(x)$ by solving $\Phi(\alpha^i) = 0$, $\Phi(\alpha^{i+1}) = 0$, $\Phi(\alpha^{i+2}) = 0$, $\Phi(\alpha^{i+3}) = 0$

In this problem, after applying the Φ : $\Phi(\alpha^4) = 0 \Leftrightarrow \alpha^4 = 1 + \alpha$
Thus for the 1st part of the circuit :

$$\begin{aligned}\alpha^1\beta &= \alpha^1 (b_0 + b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3) \\ &= b_0\alpha^1 + b_1\alpha^2 + b_2\alpha^3 + b_3\alpha^4 \\ &= b_0\alpha + b_1\alpha^2 + b_2\alpha^3 + b_3(1+\alpha) \\ &= b_3 + (b_0 + b_3)\alpha + b_1\alpha^2 + b_2\alpha^3\end{aligned}$$

Therefore the 1st part of the circuit, which the connection is the coefficient of α^i is



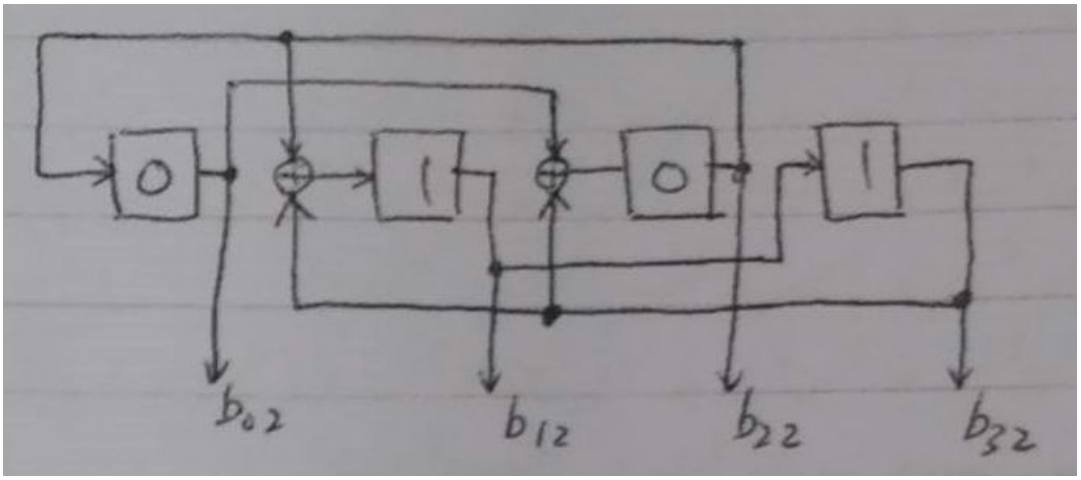
Now the second part of the circuit :

$$\begin{aligned}\alpha^2\beta &= \alpha^2 (b_0 + b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3) \\ &= b_0\alpha^2 + b_1\alpha^3 + b_2\alpha^4 + b_3\alpha^5\end{aligned}$$

Since $\Phi(\alpha^4) = 0 \Leftrightarrow \alpha^4 = 1 + \alpha$, so $\Leftrightarrow \alpha^5 = \alpha + \alpha^2$

$$\begin{aligned}&= b_0\alpha^2 + b_1\alpha^3 + b_2(1+\alpha) + b_3(\alpha + \alpha^2) \\ &= b_2 + (b_2 + b_3)\alpha + (b_0 + b_3)\alpha^2 + b_1\alpha^3\end{aligned}$$

So the diagram is :

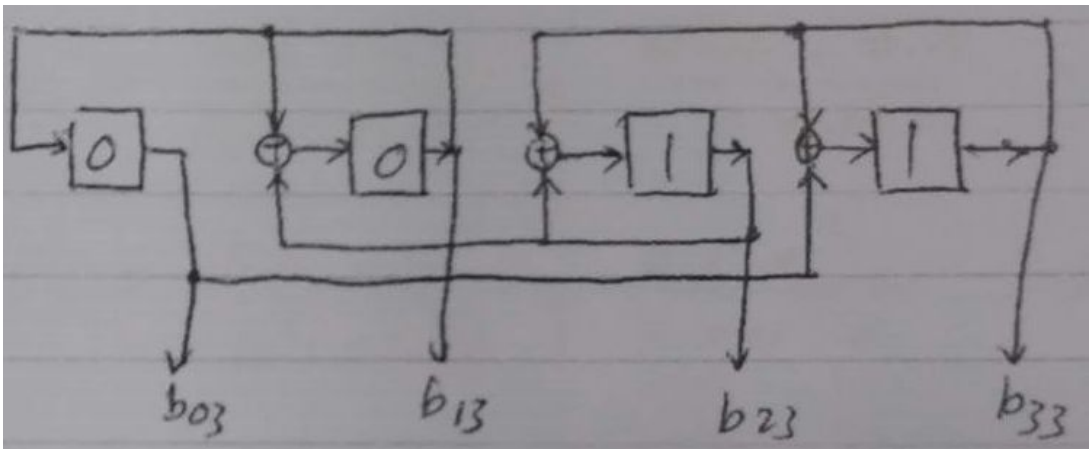


Now the third part of the circuit :

$$\alpha^3\beta = \alpha^3 (b_0 + b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3)$$
$$= b_0\alpha^3 + b_1\alpha^4 + b_2\alpha^5 + b_3\alpha^6$$

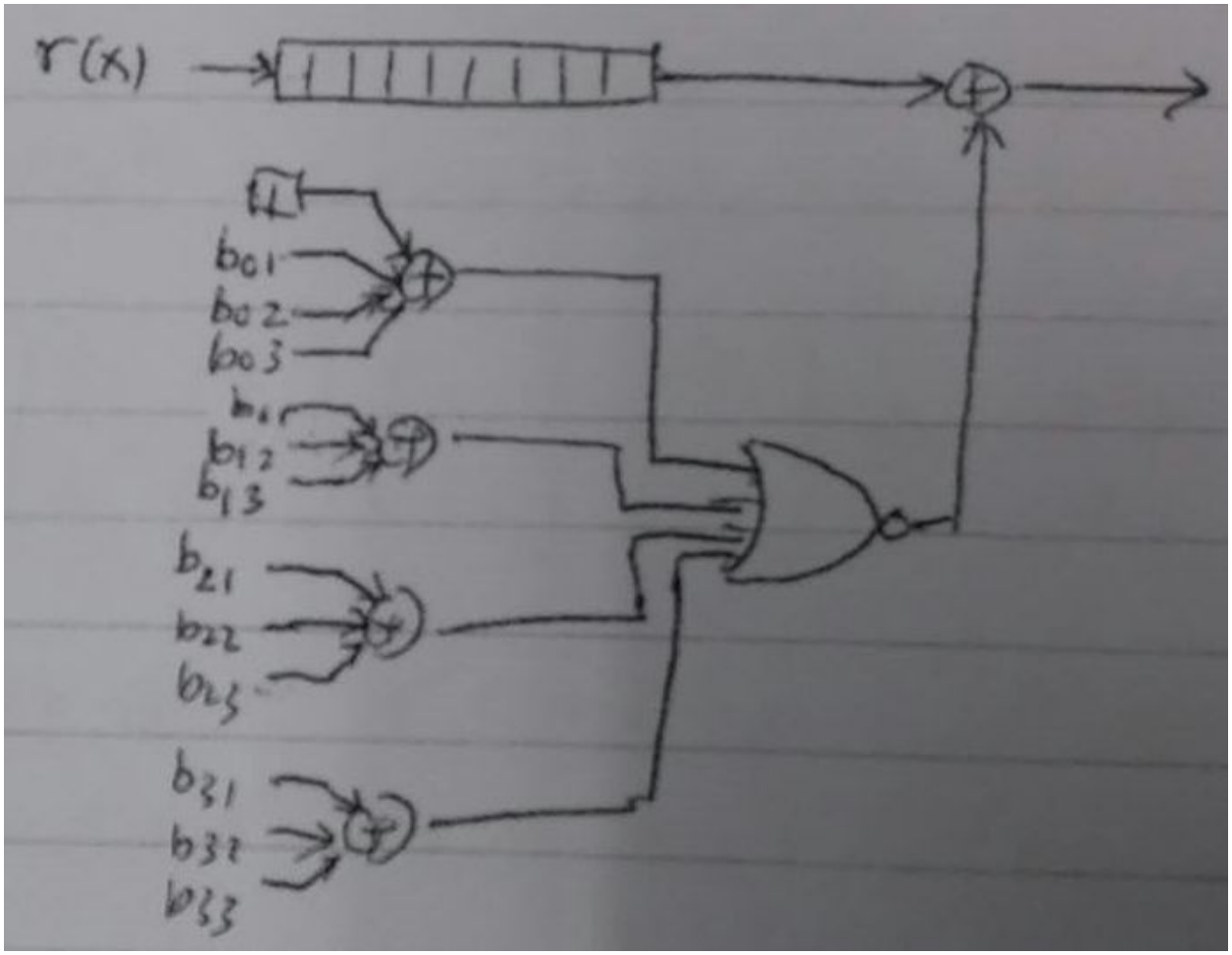
$\alpha^4 = 1 + \alpha \;\Leftrightarrow\; \alpha^5 = \alpha + \alpha^2 \;\Leftrightarrow\; \alpha^6 = \alpha^2 + \alpha^3$

$$= b_0\alpha^3 + b_1(1 + \alpha) + b_2(\alpha + \alpha^2) + b_3(\alpha^2 + \alpha^3)$$
$$= b_1 + (b_1 + b_2)\alpha + (b_2 + b_3)\alpha^2 + (b_0 + b_3)\alpha^3$$



The overall circuit

The overall circuit is as shown as follows



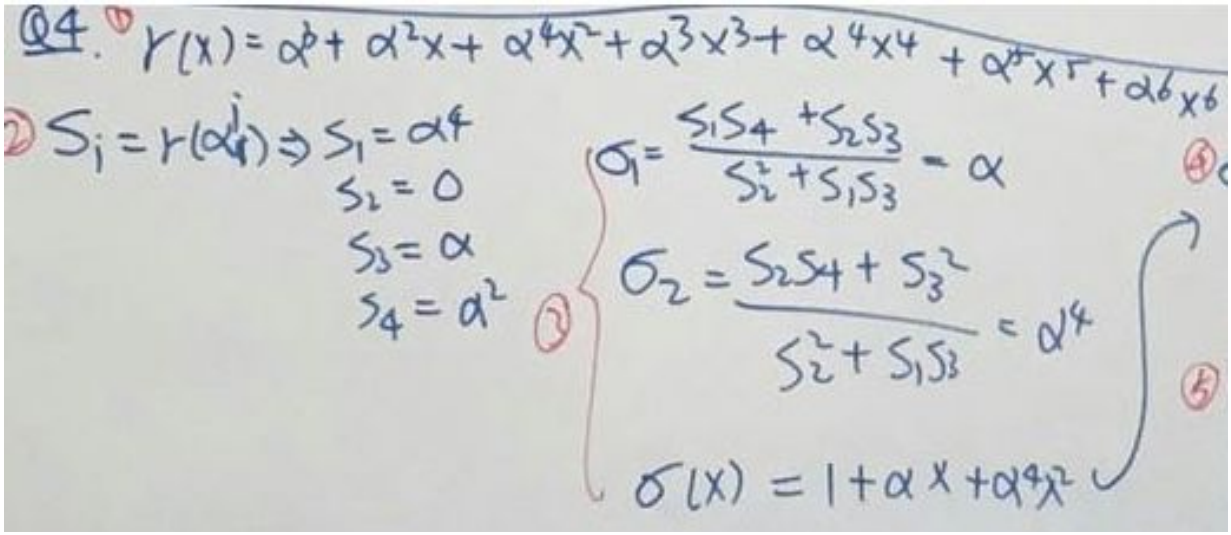
Q4. Reed Solomon Code

Step 1. Obtain the receiver polynomial $r(x)$, where the coefficients are determined by the received codeword.

Step 2. Compute the syndrome $S_i = r(\alpha^i)$, $i = 1,2,3,4$ (t=2 for this problem)

Step 3. Compute the error location polynomial as $\sigma(x) = 1 + \sigma_1 x + \sigma_2 x^2$

where $\sigma_1 = \frac{S_1 S_4 + S_2 S_3}{S_2^2 + S_1 S_3}$ and $\sigma_2 = \frac{S_2 S_4 + S_3^2}{S_2^2 + S_1 S_3}$



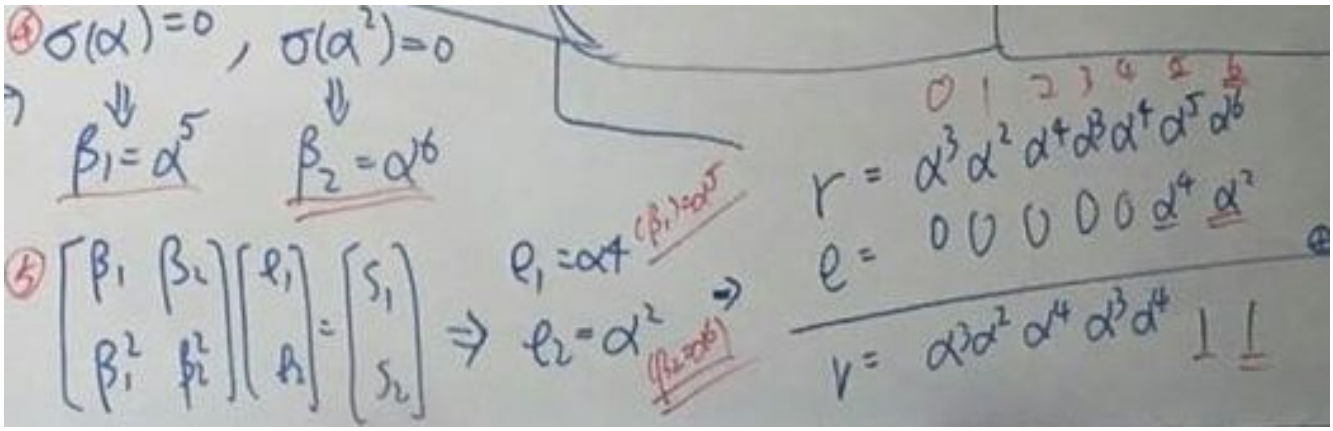
Step 4. Find the error position. After checking all the $\sigma(\alpha^i)$:

$$\sigma(\alpha^1) = 0 \quad \Leftrightarrow \beta_1 = \alpha^5$$
$$\sigma(\alpha^2) = 0 \quad \Leftrightarrow \beta_2 = \alpha^6$$

Step 5. Find the error magnitude $e_{1,2}$ by solving

$$S_1 = e_1\beta_1 + e_2\beta_2$$
$$S_2 = e_1\beta_1^2 + e_2\beta_2^2$$

Step 6. Correct the received codeword



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