Prove $\|X\|_F \geq \|X\|_2$

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Frobenius Norm

Given a real (complex) n-by-n square matrices X, the **Frobenius Norm** of X is the sum of the (absolute) squares of its elements of X

$$\|X\|_F = \sqrt{\sum_{ij} X_{ii}^2}.$$

- It can be considered as vector norm

$$\|X\|_F = \|\text{vec}(X)\|_2$$

- It can be defined by trace

$$\|X\|_F = \sqrt{\mathrm{Tr}(X^\top X)}$$

- It can be defined by singular value

$$\|X\|_F = \sqrt{\sum_i \sigma_i^2(X)}$$

Spectral Norm

Given a real (complex) n-by-n square matrices X, the **Spectral Norm** of X is the square root of the largest eigenvalue of $X^\top X$.

$$\|X\|_2 = \sqrt{\lambda_1(X^\top X)} = \sqrt{\sigma_1(X)}.$$

- If X is complex, transpose become conjugate transpose.
- It is induced by the vector L_2 norm

$$\|X\|_2 = \max_{v \neq 0} \frac{\|Xv\|_2}{\|v\|_2}$$

Inequality between $\|X\|_F$ and $\|X\|_2$

Given a real (complex) n-by-n square matrices X,

$$\|X\|_F \leq \|X\|_2.$$

How to prove :

- Proof using eigenvalue
- Proof using inequality

Proof using eigenvalue

Tools we need :

- Trace of a matrix equal to the sum of all eigenvalues of that matrix.

$$\mathrm{Tr}(X) = \sum_i \lambda_i(X)$$

- $X^\top X$ is a positive semi-definite
- Eigenvalues of positive semi-definite matrix are all non-negative

$$\lambda_i(X^\top X) \geq 0$$

Hence

$$\begin{aligned}
\|X\|_F &= \sqrt{\mathrm{Tr}(X^\top X)} \\
&= \sqrt{\sum_i \lambda_i(X^\top X)} \\
&= \sqrt{\lambda_1(X^\top X) + \underbrace{\lambda_2(X^\top X) + \lambda_3(X^\top X) + \dots}_{\geq 0}} \\
&\geq \sqrt{\lambda_1(X^\top X)} \\
&= \|X\|_2 \quad \square
\end{aligned}$$

Proof using inequality

Tools we need :

- Triangle inequality $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$
- Norm inequality with scalar multiplication $\|cX\|_2 \leq |c|\|X\|_2$
- Cauchy-Schwartz inequality $\left(\sum_i a_i b_i\right)^2 \leq \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right)$

Start with $\|Xv\|_2^2$ with v normalized : $v = \sum_i c_i e_i$ that $\sum_i |c_i|^2 = 1$

$$\begin{aligned}
\|Xv\|_2^2 &= \left\| X\left(\sum_i c_i e_i\right)\right\|_2^2 = \left\|\sum_i c_i X e_i\right\|_2^2 \\
\text{(Triangle inequality)} &\leq \sum_i \|c_i X e_i\|_2^2 \\
\text{(norm inequality)} &\leq \sum_i |c_i| \|X e_i\|_2^2 \\
\text{(C.S. inequality)} &\leq \underbrace{\left(\sum_i |c_i|^2\right)}_{=1} \underbrace{\left(\sum_i \|X e_i\|_2^2\right)}_{\|X\|_F^2} \quad \square
\end{aligned}$$