# Get rid of the Hadamard product in $\|\omega \circ (\mathbf{A}\mathbf{x} - \mathbf{b})\|$

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First draft : June 23, 2019
Last update : October 28, 2019

# Least square problem

**Least square** : given $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, find $\mathbf{x} \in \mathbb{R}^n$ by solving

$$(\mathcal{P}_1) \ : \ \min_{\mathbf{x}} f_1(\mathbf{x}) = \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2.$$

**Weighted least square** : Suppose we are given $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\omega \in \mathbb{R}^m$, find $\mathbf{x} \in \mathbb{R}^n$ by solving

$$(\mathcal{P}_2) \ : \ \min_{\mathbf{x}} f_2(\mathbf{x}) = \frac{1}{2}\|\omega \circ (\mathbf{A}\mathbf{x} - \mathbf{b})\|_2^2$$

where $\circ$ is Hadamard (element-wise) product, and $\omega$ is the weight.

How to solve $(\mathcal{P}_2)$ in a similar fashion as $(\mathcal{P}_1)$?
e.g., if we use gradient descent with step size tuned as the inverse of the Lipschitz constant of $\nabla f(\mathbf{x})$, we know that

$$L(f_1) = \|\mathbf{A}\|_2.$$

But what is $L(f_2)$?

# The equivalence between vector-vector Hadamard product and diagonal-matrix-vector product

To get rid of element-wise operator $\circ$, one has to use the equivalence between vector-vector Hadamard product and diagonal-matrix-vector product.

$$\omega \circ \mathbf{x} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix} \circ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \omega_1 x_1 \\ \omega_2 x_2 \\ \vdots \\ \omega_n x_n \end{bmatrix} = \begin{bmatrix} \omega_1 & 0 & \dots & 0 \\ 0 & \omega_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{D}\mathbf{x}$$

where the diagonal matrix $\mathbf{D}$ is defined as

$$\mathbf{D}_{ii} = \omega_i \quad \text{for all } i$$

# The weighted least square problem

We can now remove $\circ$ in $(\mathcal{P}_2)$.

Define a $m$-by-$m$ matrix $\mathbf{D}$ that $D_{ii} = \omega_i$ for all $i$.

Then we have

$$(\mathcal{P}_2') \;:\; \min_{\mathbf{x}} f_2'(\mathbf{x}) = \frac{1}{2}\|\mathbf{D}(\mathbf{A}\mathbf{x} - \mathbf{b})\|_2^2 = \frac{1}{2}\|\mathbf{D}\mathbf{A}\mathbf{x} - \mathbf{D}\mathbf{b}\|_2^2.$$

Hence now the Lipschitz constant of $\nabla f_2'(\mathbf{x})$ will be

$$L(f_2) = \|\mathbf{D}\mathbf{A}\|_2$$

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