Quaternion

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What is quaternion

Recall that

- A real number $a \in \mathbb{R}$ is geometrically represented by a point on the real number line
- A complex number $a + bi \in \mathbb{C}$ is geometrically represented by a point in the two-dimensional complex plan
- A real number is like a 1-dimensional number and a complex number is like a 2-dimensional number

A quaternion is a 4-dimensional number in the form

$$q = q_0 + q_1 i + q_2 j + q_3 k, \quad \text{where} \quad q_{0,1,2,3} \in \mathbb{R}$$

The set of quaternion is denoted by $\mathbb{H}$, to remember William **H**amilton who discover quaternion in 1843.
Note : Olinde Rodrigues described it in 1840.

Why quaternion

Recall why complex number

- Complex number is very useful in describing two-dimensional the rotation on a plane
- Given two complex numbers $x = x_0 + x_1 i, y = y_0 + y_1 i$, the multiplication xy describe the process of "rotation and scaling"
- Let $z = z_0 + z_1 i$ such that $z = xy$, then we have

$$\text{modulus}(z) = |z| = |x||y| \;\; \text{and} \;\; \text{argument}(z) = \angle z = \angle x + \angle y,$$

  which is basically the result of Euler's formula

$$|z|e^{i\theta_z} = z = xy = |x|e^{i\theta_x}|y|e^{i\theta_y} = |x||y|e^{i(\theta_x+\theta_y)}$$

Then why quaternion : quaternion is very useful in describing rotation in three-dimension $\implies$ lots of applications in robotics.

One slide summary of quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$

- Real part of q is $\mathrm{R}(q) = q_0$
- Imaginary part of q is $\mathrm{I}(q) = q_1 i + q_2 j + q_3 k$, or $\mathbf{q} = q_1 i + q_2 j + q_3 k$
- q is called purely imaginary if $q_0 = 0$
- $q = \mathrm{R}(q) + \mathrm{I}(q)$, or $q = q_0 + \mathbf{q}$
- $i^2 = j^2 = k^2 = ijk = -1$
- $ij = -ji = k, jk = -kj = i, ki = -ik = j$
- Given $p, q \in \mathbb{H}$, $p + q = p_0 + q_0 + \mathbf{p} + \mathbf{q}$
- Given $p, q \in \mathbb{H}$, $pq = p_0 q_0 + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{pq}$ where $\mathbf{pq} = \mathbf{p} \times \mathbf{q} - \mathbf{p} \cdot \mathbf{q}$ (cross product minus dot product). Or, after expansion

$$\begin{aligned} pq &= (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3) + (p_0 q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2) i \\ &\quad + (p_0 q_2 - p_1 q_3 + p_2 q_0 + p_3 q_1) j + (p_0 q_3 + p_1 q_2 - p_2 q_1 + p_3 q_0) k \end{aligned}$$

- Quaternion multiplication is not commutative
- Conjugate $q^* = q_0 - q_1 i - q_2 j - q_3 k$
- Modulus $|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$
- Inverse $q^{-1} = \frac{q^*}{|q|^2}$, $|q| \neq 0$

Modern use of quaternion in science and engineering

Classically, quaternion is used as a tool to represent rotation in 3-dimensional space, hence it has application in

- Modelling the spin of particle in physics
- Modelling of robotic arm placement, which is an important topic in robotic mechanics
- Computer Graphics on 3d animation related to rotation

The applications of quaternion in machine learning includes

- Modelling of colour images : colour image consist of Red, Green, Blue channel can be modelled as a pure quaternion. For example, an image of size m-by-n can be modelled as

  $$M = M_0 + M_B i + M_G j + M_B k,\ M_0 = 0$$

  In this sense, the operation on the quaternion matrix preserves the structure of each colour channel, instead of mixing the information of the channels in the classical way of colour image processing

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