Singular value thresholding operator is the solution to $\arg\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{X}-\mathbf{Y}\|_F^2 + \tau\|\mathbf{X}\|_*$

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The minimization problem

Given a matrix $\mathbf{Y} \in \mathbb{R}^{m \times n}$, find the matrix $\mathbf{X}$ by solving the following optimization problem $(\mathcal{P})$:

$$(\mathcal{P}) \;\; : \;\; \mathbf{X} = \arg\min_{\mathbf{X}} f(\mathbf{X}) = \frac{1}{2}\|\mathbf{X} - \mathbf{Y}\|_F^2 + \tau\|\mathbf{X}\|_*,$$

where

- $\mathbf{X}$ is the optimization variable in $\mathbb{R}^{m \times n}$
- $f(\mathbf{X}) = \frac{1}{2}\|\mathbf{X} - \mathbf{Y}\|_F^2 + \tau\|\mathbf{X}\|_*$ is the objective function :
  - τ is the regularization parameter in $\mathbb{R}$
  - $\|\cdot\|_F$ is the Frobenius norm
  - $\|\cdot\|_*$ is the Ky Fan norm (the Nuclear norm), defined as the sum of singular values :

$$\|\mathbf{A}\|_* = \sum_i |\sigma_i(\mathbf{A})| = \sum_i \sigma_i(\mathbf{A}).$$

  - note that $\|\cdot\|_F$ is smooth (differentiable) but $\|\cdot\|_*$ is not

Singular value thresholding operator

For any matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, we have

$$\mathbf{A} \overset{\text{SVD}}{=} \mathbf{U}\Sigma\mathbf{V}^\top = \mathbf{U}\mathsf{Diag}\big(\sigma_i(\mathbf{A})\big)\mathbf{V}^\top.$$

Define the thresholding operator $\mathcal{D}_a$ of Σ as follows :

$$\mathcal{D}_a(\Sigma) = \mathsf{Diag}\big([\sigma_i - a]_+\big) = \mathsf{Diag}\big(\max\{\sigma_i(\mathbf{A}) - a, 0\}\big).$$

We define the *Singular value thresholding* (SVT) of a matrix $\mathbf{A}$ as

$$\mathsf{SVT}_a(\mathbf{A}) = \mathbf{U}\mathcal{D}_a(\Sigma)\mathbf{V}^\top.$$

What it does : subtract all the singular value of $\mathbf{A}$ by the parameter a, if the result after the subtraction is negative, replace it by zero.

Note : if all the singular values of $\mathbf{A}$ are larger than a, then we can drop the max operator and write it as

$$\mathsf{SVT}_a(\mathbf{A}) = \mathbf{U}(\Sigma - a\mathbf{I})\mathbf{V}^\top.$$

As singular values of any matrix are always non-negative, in other words what SVT does is shrinks the singular values of $\mathbf{A}$ towards zero by a amount.

Theorem relating the SVT operator and the problem $(\mathcal{P})$

Theorem† The SVT operator

$$\mathsf{SVT}_\tau(\mathbf{Y}) = \mathbf{U}\mathcal{D}_\tau(\Sigma)\mathbf{V}^\top,$$

is the solution to the minimization problem

$$(\mathcal{P}) \ : \ \mathbf{X} = \arg\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{X}-\mathbf{Y}\|_F^2 + \tau\|\mathbf{X}\|_*.$$

That is, we have the equality

$$\mathsf{SVT}_\tau(\mathbf{Y}) = \arg\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{X}-\mathbf{Y}\|_F^2 + \tau\|\mathbf{X}\|_*.$$

This document : proof this.

† Reference : Cai, Jian-Feng, Emmanuel Candés, and Zuowei Shen. "A singular value thresholding algorithm for matrix completion." *SIAM Journal on Optimization*, 20.4 (2010): 1956-1982.

The pre-requisites to understand the proof

- Sub-gradient characterization of the first order optimality condition for convex (non-smooth) function
  - Sub-gradient. For convex (but not necessarily smooth function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$, $\mathbf{Z}$ is a sub-gradient of f at a point $\mathbf{X} = \mathbf{X}_0$, denoted as $\mathbf{Z} \in \partial f(\mathbf{X}_0)$, if
  $$f(\mathbf{X}) \geq f(\mathbf{X}_0) + \langle \mathbf{Z}, \mathbf{X} - \mathbf{X}_0 \rangle$$
  is true for all $\mathbf{X}$.
  - First order optimality condition. A point $\mathbf{X}_0$ minimizes $f(\mathbf{X})$ if and only if $\mathbf{0}$ is a sub-gradient of f at $\mathbf{X}_0$. That is,
  $$\mathbf{0} \in \mathbf{X}_0 - \mathbf{Y} + \tau\partial\|\mathbf{X}_0\|_*, \tag{1}$$
- Ky Fan norm (nuclear norm) $\|\mathbf{A}\|_*$
  - Ky Fan norm is a norm and hence is convex. See the proof here.
  - The sub-differential of $\|\cdot\|*$ for a matrix $\mathbf{A} \overset{\text{SVD}}{=} \mathbf{U}\Sigma\mathbf{V}^\top$:
  $\partial\|\mathbf{A}\|_* = \left\{\mathbf{U}\mathbf{V}^\top + \mathbf{W} \,\middle|\, \mathbf{W} \in \mathbb{R}^{m\times n}, \mathbf{U}^\top\mathbf{W} = \mathbf{0}, \mathbf{W}\mathbf{V} = \mathbf{0}, \|\mathbf{W}\|_2 \leq 1\right\}.$

    Proof : G.A. Watson, "Characterization of the subdifferential of some matrix norms", Linear Algebra and its Applications, vol.170, pp33-45, 1992

Notice that

$$(\mathcal{P}) \;\;:\;\; \arg\min_{\mathbf{X}} f(\mathbf{X}) = \frac{1}{2}\|\mathbf{X}-\mathbf{Y}\|_F^2 + \tau\|\mathbf{X}\|_*.$$

has a unique minimizer as the objective function f is a strictly convex function :

- $\|\mathbf{X}-\mathbf{Y}\|_F^2$ is strictly convex in $\mathbf{X}$
- $\|\mathbf{X}\|_*$ is a norm so it is convex
- Strictly convex function + convex function = strictly convex function

So what we need to do : show the global minimizer $\mathbf{X}^*$ of $(\mathcal{P})$ equals to $\text{SVT}_\tau(\mathbf{Y})$. To do that, we use the first-order optimality condition (FOC) for f. Since f contains non-differentiable part, so we use the sub-gradient characterization of FOC.

What we have :

$$f(\mathbf{X}) = \frac{1}{2}\|\mathbf{X} - \mathbf{Y}\|_F^2 + \tau\|\mathbf{X}\|_*.$$

The sub-gradient characterization of FOC for f : a point $\mathbf{X}_0$ minimizes $f(\mathbf{X})$ if and only if $\mathbf{0}$ is a sub-gradient of f at $\mathbf{X}_0$:

$$\mathbf{0} \in \mathbf{X}_0 - \mathbf{Y} + \tau\partial\|\mathbf{X}_0\|_*.$$

Re-write it into a "better expression" :

$$\mathbf{Y} - \mathbf{X}_0 \in \tau\partial\|\mathbf{X}_0\|_*.$$

i.e. to prove the theorem, we have to show that, for $\mathbf{X}_0 =\text{SVT}_\tau(\mathbf{Y})$, it minimizes $f(\mathbf{X})$ if and only if you subtract $\mathbf{X}_0$ from $\mathbf{Y}$, the result belongs to the set of sub-gradient of the Ky Fan norm multiplied by the constant τ.

As we need to show the inclusion, we need to know the sub-differential (set of sub-gradients) $\partial\|\mathbf{X}_0\|_*$.

The sub-differential of $\|\mathbf{X}\|_*$ is the following set

$$\partial\|\mathbf{X}\|_* = \left\{\mathbf{U}\mathbf{V}^\top + \mathbf{W} \,\middle|\, \mathbf{W} \in \mathbb{R}^{m\times n}, \mathbf{U}^\top\mathbf{W} = \mathbf{0}, \mathbf{W}\mathbf{V} = \mathbf{0}, \|\mathbf{W}\|_2 \leq 1\right\},$$

where $\mathbf{U}, \mathbf{V}$ are obtained from the SVD of the matrix $\mathbf{X}$.

Therefore, what we have to do, is to show, for $\mathbf{X}_0 = \mathsf{SVT}_\tau(\mathbf{Y})$, we have

$$\mathbf{Y} - \mathbf{X}_0 \in \tau\left\{\mathbf{U}\mathbf{V}^\top + \mathbf{W} \,\middle|\, \mathbf{W} \in \mathbb{R}^{m\times n}, \mathbf{U}^\top\mathbf{W} = \mathbf{0}, \mathbf{W}\mathbf{V} = \mathbf{0}, \|\mathbf{W}\|_2 \leq 1\right\}.$$

And the essence of the proof is to make use of this inclusion relation :

- obtain the expression $\mathbf{Y} - \mathbf{X}_0 = \mathbf{Y} - \mathsf{SVT}_\tau(\mathbf{Y})$
- show that $\mathbf{Y} - \mathsf{SVT}_\tau(\mathbf{Y})$ is inside the set

1. Obtain the expression $\mathbf{Y}-$ $\textbf{SVT}_\tau(\mathbf{Y})$

Split the SVD $\mathbf{Y} = \mathbf{U}\Sigma\mathbf{V}^\top$ into two groups based on the sign of $\sigma_i - \tau$:

$$\mathbf{Y} = \mathbf{U}_{>\tau}\Sigma_{>\tau}\mathbf{V}_{>\tau}^\top + \mathbf{U}_{\le\tau}\Sigma_{\le\tau}\mathbf{V}_{\le\tau}^\top.$$

Here the SVT operator with parameter τ kills the second term of $\mathbf{Y}$:

$$\begin{aligned}
\mathsf{SVT}_\tau(\mathbf{Y}) &= \mathsf{SVT}_\tau(\mathbf{U}_{>\tau}\Sigma_{>\tau}\mathbf{V}_{>\tau}^\top + \mathbf{U}_{\le\tau}\Sigma_{\le\tau}\mathbf{V}_{\le\tau}^\top) \\
&= \mathsf{SVT}_\tau(\mathbf{U}_{>\tau}\Sigma_{>\tau}\mathbf{V}_{>\tau}^\top) + \mathsf{SVT}_\tau(\mathbf{U}_{\le\tau}\Sigma_{\le\tau}\mathbf{V}_{\le\tau}^\top) \\
&= \mathbf{U}_{>\tau}\mathcal{D}_\tau(\Sigma_{>\tau})\mathbf{V}_{>\tau}^\top + \mathbf{U}_{\le\tau}\underbrace{\mathcal{D}_\tau\left(\Sigma_{\le\tau}\right)}_{0}\mathbf{V}_{\le\tau}^\top \\
&= \mathbf{U}_{>\tau}\mathcal{D}_\tau(\Sigma_{>\tau})\mathbf{V}_{>\tau}^\top \\
&= \mathbf{U}_{>\tau}\underbrace{\mathsf{Diag}(\max\{\sigma_i - \tau, 0\})}_{\text{all are positive}}\mathbf{V}_{>\tau}^\top \\
&= \mathbf{U}_{>\tau}(\Sigma_{>\tau} - \tau\mathbf{I})\mathbf{V}_{>\tau}^\top.
\end{aligned}$$

Hence, we have

$$\mathbf{Y} - \mathsf{SVT}_\tau(\mathbf{Y}) = \mathbf{U}_{\le\tau}\Sigma_{\le\tau}\mathbf{V}_{\le\tau}^\top + \tau\mathbf{U}_{>\tau}\mathbf{V}_{>\tau}^\top.$$

2. Show $\mathbf{Y}-$ SVT$_\tau(\mathbf{Y}) \in \tau\partial\|\mathbf{X}\|_*$

Let $\mathbf{W} = \tau^{-1}\mathbf{U}_{\leq\tau}\Sigma_{\leq\tau}\mathbf{V}_{\leq\tau}^\top$, then we have

$$\begin{aligned}\mathbf{Y} - \mathsf{SVT}_\tau(\mathbf{Y}) &= \mathbf{U}_{\leq\tau}\Sigma_{\leq\tau}\mathbf{V}_{\leq\tau}^\top + \tau\mathbf{U}_{>\tau}\mathbf{V}_{>\tau}^\top, \\ &= \tau(\mathbf{W} + \mathbf{U}_{>\tau}\mathbf{V}_{>\tau}^\top)\end{aligned}$$

Hence, we have to show $\mathbf{W} + \mathbf{U}_{>\tau}\mathbf{V}_{>\tau}^\top$ is inside the set

$$\left\{\mathbf{U}\mathbf{V}^\top + \mathbf{W} \,\middle|\, \mathbf{W} \in \mathbb{R}^{m\times n}, \mathbf{U}^\top\mathbf{W} = \mathbf{0}, \mathbf{W}\mathbf{V} = \mathbf{0}, \|\mathbf{W}\|_2 \leq 1\right\}.$$

by checking the three inclusion conditions :

- Does $\|\mathbf{W}\|_2 \leq 1$?
- Does $\mathbf{U}_{>\tau}^\top\mathbf{W} = \mathbf{0}$?
- Does $\mathbf{W}\mathbf{V}_{>\tau} = \mathbf{0}$?

The proof of the theorem . . . 6/6

For $\mathbf{W} = \tau^{-1}\mathbf{U}_{\leq\tau}\Sigma_{\leq\tau}\mathbf{V}_{\leq\tau}^{\top}$,

- Does $\|\mathbf{W}\|_2 \leq 1$?
  YES : $\|\mathbf{W}\|_2 = \tau^{-1}\|\Sigma_{\leq\tau}\|_2 \leq 1$.
- Does $\mathbf{U}^{\top}\mathbf{W} = \mathbf{0}$?
  YES : as columns of $\mathbf{U}$ are orthonormal to each other, so
  $$\mathbf{U}_{>\tau}^{\top}\mathbf{W} = \tau^{-1}\underbrace{\mathbf{U}_{>\tau}^{\top}\mathbf{U}_{\leq\tau}}_{\mathbf{0}}\Sigma_{\leq\tau}\mathbf{V}_{\leq\tau}^{\top} = \mathbf{0}$$
- Does $\mathbf{W}\mathbf{V} = \mathbf{0}$?
  YES : as columns of $\mathbf{V}$ are orthonormal to each other, so
  $$\mathbf{W}\mathbf{V} = \tau^{-1}\mathbf{U}_{>\tau}^{\top}\Sigma_{\leq\tau}\underbrace{\mathbf{V}_{\leq\tau}^{\top}\mathbf{V}_{>\tau}}_{\mathbf{0}} = \mathbf{0}$$

Hence, we showed $\mathbf{Y}-$ SVT$_\tau(\mathbf{Y}) \in \tau\partial\|\mathbf{X}\|_*$. □

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