Some functions of singular values

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Frobenius norm

- Given a matrix $\mathbf{X} \in \mathbb{R}^{m \times n}$, Frobenius norm is defined as

$$\|\mathbf{X}\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |X_{ij}|^2}.$$

It is the square root of the sum of the absolute squares of its elements

- In short hand notation :

$$\|\mathbf{X}\|_F = \sqrt{\sum_{ij} |X_{ij}|^2}$$

As $x^2 = |x|^2$, so in even short hand notation

$$\|\mathbf{X}\|_F = \sqrt{\sum_{ij} X_{ij}^2}$$

Although short hand notations are compact, but they may let people forget the absolute value sign : if $\mathbf{X} \in \mathbb{C}^{m \times n}$, then X_{ij} are complex valued and $|X_{ij}|^2 \neq X_{ij}^2$.

Frobenius norm as a function of singular values

- Frobenius norm squared is $\|\mathbf{X}\|_F^2 = \|\mathbf{X}\|_F = \sum_{ij} |X_{ij}|^2$.
- Recall the trace of a matrix is $\mathrm{Tr}\left(\mathbf{X}\right) = \sum X_{ii}$.
- Frobenius norm squared is linked with trace

$$\|\mathbf{X}\|_F^2 = \sum_{ij} |X_{ij}|^2 = \sum_i \left(\sum_j X_{ij}^T X_{ji} \right) = \sum_i (X^T X)_{ii} = \mathrm{Tr}\left(\mathbf{X}^\top \mathbf{X}\right).$$

- Frobenius norm is the L_2-norm of singular vector
  - Let $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$
  - $\mathbf{X}^\top \mathbf{X} = \mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^\top$
  - $\mathrm{Tr}\left(\mathbf{X}^\top \mathbf{X}\right) = \mathrm{Tr}\left(\mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^\top\right)$
  - Trace is invariant under cyclic permutations : $\mathrm{Tr}\left(\mathbf{ABC}\right) = \mathrm{Tr}\left(\mathbf{BCA}\right)$.
  - $\mathrm{Tr}\left(\mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^\top\right) = \mathrm{Tr}\left(\mathbf{\Sigma}^2\mathbf{V}^\top\mathbf{V}\right) = \mathrm{Tr}\left(\mathbf{\Sigma}^2\right)$

  Therefore,

$$\|\mathbf{X}\|_F^2 = \mathrm{Tr}\left(\mathbf{X}^\top \mathbf{X}\right) = \mathrm{Tr}\left(\mathbf{\Sigma}^2\right) = \sum_{i=1} \sigma_i^2 = \|\boldsymbol{\sigma}\|_2^2.$$

Operator norm, Nuclear norm and rank

- Operoator norm (spectral norm / 2-norm)

$$\|\mathbf{X}\|_{\text{op}} = \sup_{\|\mathbf{b}\|_2 \leq 1} \|\mathbf{X}\mathbf{b}\|_2 = \max_i \sigma_i = \|\boldsymbol{\sigma}\|_\infty.$$

- Nuclear norm (KyFan norm)

$$\|\mathbf{X}\|_* = \sum_i |\sigma_i| = \|\boldsymbol{\sigma}\|_1.$$

- Matrix rank, which is not a matrix norm, it

$$\text{rank}(\mathbf{X}) = \|\boldsymbol{\sigma}\|_0.$$

Determinant and log-determinant

- Determinant

$$\det(\mathbf{X}) = \prod_i \sigma_i.$$

- Determinant of Gramian plus perturbation

$$\det(\mathbf{X}^\top \mathbf{X} + \delta \mathbf{I}) = \prod_i (\sigma_i^2 + \delta).$$

- log-determinant

$$\log \det(\mathbf{X}) = \sum_i \log \sigma_i.$$

- log-determinant of Gramian plus perturbation

$$\log \det(\mathbf{X}^\top \mathbf{X}) = \sum_i \log(\sigma_i^2 + \delta).$$

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Table: List of functions of singular values

Function in matrix $\mathbf{X}$	Function of singular values
$\|\mathbf{X}\|_F^2$	$\|\boldsymbol{\sigma}\|_2^2$
$\|\mathbf{X}\|_{\mathrm{op}}$	$\|\boldsymbol{\sigma}\|_\infty$
$\|\mathbf{X}\|_*$	$\|\boldsymbol{\sigma}\|_1$
$\mathrm{rank}(\mathbf{X})$	$\|\boldsymbol{\sigma}\|_0$
$\det(\mathbf{X})$	$\prod_i \sigma_i$
$\log\det(\mathbf{X})$	$\sum_i \log \sigma_i$

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