$\mathsf{Tr}(A^2) \geq \frac{(\mathsf{Tr}A)^2}{n}$

Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive semi-definite and $\mathsf{rank}(A) = n$ (i.e. A is full rank). Then

$$\mathsf{Tr}(A^2) \geq \frac{(\mathsf{Tr}A)^2}{n}$$

Proof : let λ be the eigenvalues of A,

$$\begin{aligned} \mathsf{Tr}A^2 &= \sum_i \lambda(A^2) \\ \Big(\because \lambda(A^2) = \lambda^2(A)\Big) &= \sum_i \lambda^2(A) \end{aligned}$$

As A is psd, λ are all non-negative, we can use Cauchy Schwarz inequality $\big(\sum_i a_i b_i\big)^2 \leq \big(\sum_i a_i^2\big)\big(\sum_i b_i^2\big)$. Put put $a_i = \lambda$ and $b_i = 1$ we have

$$\Big(\sum_i \lambda^2(A)\Big) \geq \frac{\big(\sum_i \lambda(A)\big)^2}{n},$$

which is $\mathsf{Tr}(A^2) \geq \frac{(\mathsf{Tr}A)^2}{n}$. $\square$

$\text{Tr}(A^2) \le (\text{Tr}(A))^2$

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As A is psd, λ are all non-negative, we can use Cauchy Schwarz inequality $\big(\sum_i a_i b_i\big)^2 \le \big(\sum_i a_i^2\big)\big(\sum_i b_i^2\big)$. Put put $a_i = b_i = \lambda$ we have

$$\Big(\sum_i \lambda^2(A)\Big)^2 \le \big(\sum_i \lambda^2(A)\big)^2,$$

which is $\text{Tr}(A^2) \ge \dfrac{(\text{Tr}A)^2}{n}$. □