# On the rank of the matrix equation $\mathbf{X} = \mathbf{YQ}$

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# Setting and a small lemma

- Matrix equation $\mathbf{X} = \mathbf{Y}\mathbf{Q}$, where $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{m \times r}$ and $\mathbf{Q} \in \mathbb{R}^{r \times r}$.
- A small lemma : for $r \leq m$, if $\mathbf{X}$ is full rank, then
  - $\mathbf{Y}$ and $\mathbf{Q}$ are full rank.
  - $\mathbf{Q}^{-1}$ exists and we have $\mathbf{Y} = \mathbf{X}\mathbf{Q}^{-1}$.

Simple proof :

- From $\mathbf{X} = \mathbf{Y}\mathbf{Q}$, we have $\operatorname{rank}(\mathbf{X}) = \operatorname{rank}(\mathbf{Y}\mathbf{Q})$.
- An elementary rank inequality

$$\operatorname{rank}(\mathbf{A}\mathbf{B}) \leq \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}.$$

- We have

$$\operatorname{rank}(\mathbf{X}) \leq \min\{\operatorname{rank}(\mathbf{Y}), \operatorname{rank}(\mathbf{Q})\}.$$

  Hence for $r \leq m$, if $\mathbf{X}$ full rank, we have $\operatorname{rank}(\mathbf{X}) = r$, and thus both $\operatorname{rank}(\mathbf{Y})$ and $\operatorname{rank}(\mathbf{Q})$ are at least $r$.
- As $\mathbf{Y} \in \mathbb{R}^{m \times r}$ and $\mathbf{Q} \in \mathbb{R}^{r \times r}$, so both $\mathbf{Y}$ and $\mathbf{Q}$ are full rank.
- Furthermore we have $\mathbf{Q}^{-1}$ exists and thus $\mathbf{Y} = \mathbf{X}\mathbf{Q}^{-1}$.

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