$\mathrm{Tr}(I - A^{-1}) \leq \log\det(A) \leq \mathrm{Tr}(A - I)$

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For a positive definite matrix $A \in \mathbb{R}^{n \times n}$, we have

$$\mathsf{Tr}(I - A^{-1}) \leq \log\det(A) \leq \mathsf{Tr}(A - I)$$

To proof this, we need the following material:

Let $\lambda(A)$ be the eigenvalue of a matrix A, we have the following facts :

- $\lambda(A) > 0$ if A is positive definite
- $\lambda(A^{-1}) = \dfrac{1}{\lambda(A)}$
- $\lambda(A \pm I) = \lambda(A) \pm 1$
- $\mathsf{Tr}A = \sum_i \lambda_i$
- $\det A = \prod_i \lambda_i$

Using the material we have

$$
\begin{aligned}
\mathsf{Tr}(I - A^{-1}) &= \sum_i \Big(1 - \frac{1}{\lambda_i}\Big) \\
\mathsf{Tr}(A - I) &= \sum_i \Big(\lambda_i - 1\Big) \\
\log\det A &= \sum_i \log\lambda_i
\end{aligned}
$$

Hence

$$
\mathsf{Tr}(I - A^{-1}) \le \log\det(A) \le \mathsf{Tr}(A - I)
$$

is equivalent to

$$
\sum_i \Big(1 - \frac{1}{\lambda_i}\Big) \le \sum_i \log\lambda_i \le \sum_i \Big(\lambda_i - 1\Big)
$$

The proof of

$$\sum_i \left(1 - \frac{1}{\lambda_i}\right) \leq \sum_i \log \lambda_i \leq \sum_i \left(\lambda_i - 1\right)$$

is to apply a fundamental inequalities on logarithm : for $x > 0$ we have

$$1 - \frac{1}{x} \leq \log x \leq x - 1$$

As matrix A is positive definte, thus $\lambda_i > 0$ for all i and hence satisfies the condition of the inequalities on logarithm.

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