$\mathrm{Tr}(\mathbf{I} - \mathbf{A}^{-1}) \leq \log\det\mathbf{A} \leq \mathrm{Tr}(\mathbf{A} - \mathbf{I})$

Andersen Ang

Mathématique et recherche opérationnelle
UMONS, Belgium

manshun.ang@umons.ac.be Homepage: angms.science

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For a positive definite matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, we have

$$\mathsf{Tr}(\mathbf{I} - \mathbf{A}^{-1}) \leq \log\det\mathbf{A} \leq \mathsf{Tr}(\mathbf{A} - \mathbf{I}).$$

To proof this, we need the following : let $\lambda(\mathbf{A})$ be the eigenvalue of a matrix $\mathbf{A}$, then

- $\lambda(\mathbf{A}) > 0$ if $\mathbf{A}$ is positive definite
- $\lambda(\mathbf{A}^{-1}) = \dfrac{1}{\lambda(\mathbf{A})}$
- $\lambda(\mathbf{A} \pm \mathbf{I}) = \lambda(\mathbf{A}) \pm 1$
- $\mathsf{Tr}\mathbf{A} = \sum_i \lambda_i$
- $\det\mathbf{A} = \prod_i \lambda_i$

Using the material we have

$$\begin{aligned}\mathsf{Tr}(\mathbf{I}-\mathbf{A}^{-1}) &= \sum_i \left(1-\frac{1}{\lambda_i}\right)\\ \mathsf{Tr}(\mathbf{A}-\mathbf{I}) &= \sum_i \left(\lambda_i - 1\right)\\ \log\det\mathbf{A} &= \sum_i \log\lambda_i\end{aligned}$$

Hence

$$\begin{aligned}& \mathsf{Tr}(\mathbf{I}-\mathbf{A}^{-1}) \le \log\det\mathbf{A} \le \mathsf{Tr}(\mathbf{A}-\mathbf{I})\\ \iff \quad & \sum_i \left(1-\frac{1}{\lambda_i}\right) \le \sum_i \log\lambda_i \le \sum_i \left(\lambda_i - 1\right)\end{aligned}$$

To proof the last inequality, apply a fundamental inequalities on logarithm :

$$1-\frac{1}{x} \le \log x \le x-1, \text{for all } x>0$$

As $\mathbf{A}$ is positive definite, so $\lambda_i > 0$ for all i and it satisfies the condition of the inequalities on logarithm.

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