Numerical 2D PDE

A quick introduction

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Poisson's equation

- Poisson's equation in three notations:

$$\Delta\phi = f$$
$$\frac{\partial\phi}{\partial x^2} + \frac{\partial\phi}{\partial y^2} = f$$
$$\phi_{xx} + \phi_{yy} = f$$

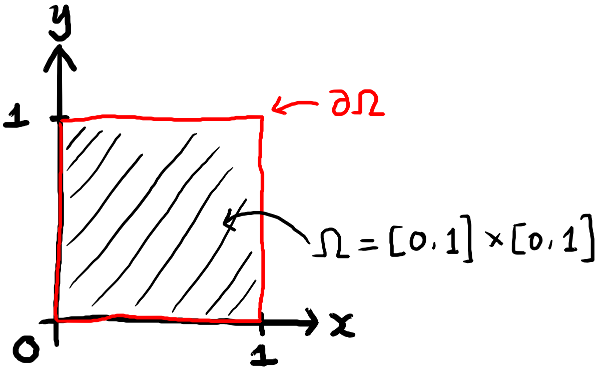
- $\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$ is the Laplace operator
- We consider two-dimensional problem in $\mathbb{R}$: both f, ϕ are functions of x and y in $\mathbb{R}$
- $f(x, y)$ is a real-valued function that represents the "source".
- The goal of solving PDE is to find the real-valued function $\phi(x, y)$.
- In general, PDE problems have no analytic solution, and most of them are solved numerically.

Boundary Value Problem (BVP)

A type of PDE problem is the BVP:

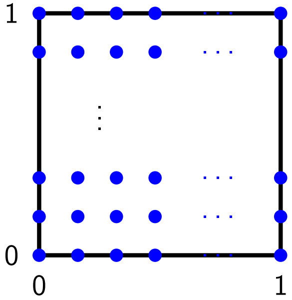
$$\text{BVP} \;:\; \begin{cases} u_{xx} + u_{yy} = f(x,y) & \forall x, y \in \Omega = [0,1] \times [0,1] \\ u(x,y) = 0 & \forall x, y \in \partial\Omega \end{cases}$$

where Ω denotes the domain and $\partial\Omega$ denotes the boundary.



Grid Discretization

- In numerical PDE, we solve the PDE problem numerically by discretizing the continuous BVP problem using a grid.



- Now we consider a uniform discretization.
  - We discretize using a rectangular grid system, more specifically, we consider the simple case of even grid size with $\Delta x = \Delta y = h$.
  - We discretize the interval [0,1] by 5 points: $x_1 = y_1 = 0$, $x_5 = y_5 = 1$ and

$$x_{i+1} = x_i + h, y_{i+1} = x_i + h, \quad h = 0.25.$$

5-point discretization example visualized

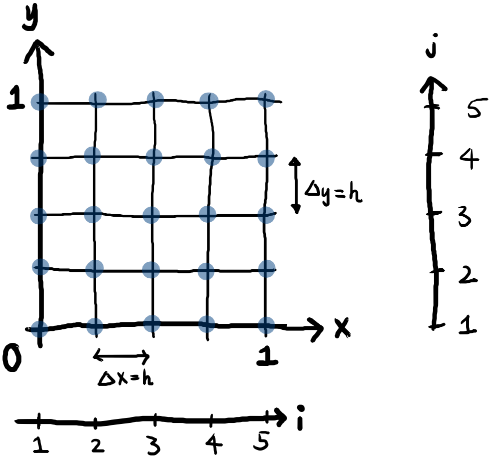
- Variable $v_{i,j}$ with

  $$i, j \in \{1, 2, 3, 4, 5\}.$$

  In total we have 25 grid points.

- Domain $\Omega = \{v_{i,j}\}$ where $i, j \in \{1, 2, 3, 4, 5\}$.

- Boundary $\partial\Omega\{v_{i,j}\}$ with $v_{i,j} = 0$ for i, j inside the union of $\Big\{(i, 1) \mid i\Big\}$, $\Big\{(i, 5) \mid j\Big\}$, $\Big\{(1, j) \mid \forall j\Big\}$, $\Big\{(5, j) \mid j\Big\}$.



Taylor series approximation

- Consider the Taylor series of u with respect to x at the point x_i

$$u(x_{i+1}, y_j) = u(x_i, y_j) + hu'(x_i, y_j) + \frac{h^2}{2!}u''(x_i, y_j) + \frac{h^3}{3!}u'''(x_i, y_j) + o(h^4). \tag{1}$$

$$u(x_{i-1}, y_j) = u(x_i, y_j) - hu'(x_i, y_j) + \frac{h^2}{2!}u''(x_i, y_j) - \frac{h^3}{3!}u'''(x_i, y_j) + o(h^4). \tag{2}$$

- (1) + (2) gives

$$u(x_{i+1}, y_j) + u(x_{i-1}, y_j) = 2u(x_i, y_j) + 2\frac{h^2}{2!}u''(x_i, y_j) + o(h^4).$$

- Rearrange: put u'' as the subject, we have

$$\begin{aligned} u''(x_i, y_j) &= \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} + o(h^4) \\ &\approx \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2}. \end{aligned}$$

- We will get a similar expression if we perform the approximation on y.

Discrete BVP

$$\text{Continuous BVP} \;:\; \begin{cases} u_{xx} + u_{yy} = f(x,y) & \forall x,y \in \Omega = [0,1]\times[0,1] \\ u(x,y) = 0 & \forall x,y \in \partial\Omega \end{cases}$$

- Let $v_{ij} \approx u(x_i, y_j)$ and let $f_{ij} \approx f(x_i, y_j)$.
  Using Taylor approximation, we obtain the Discrete BVP as

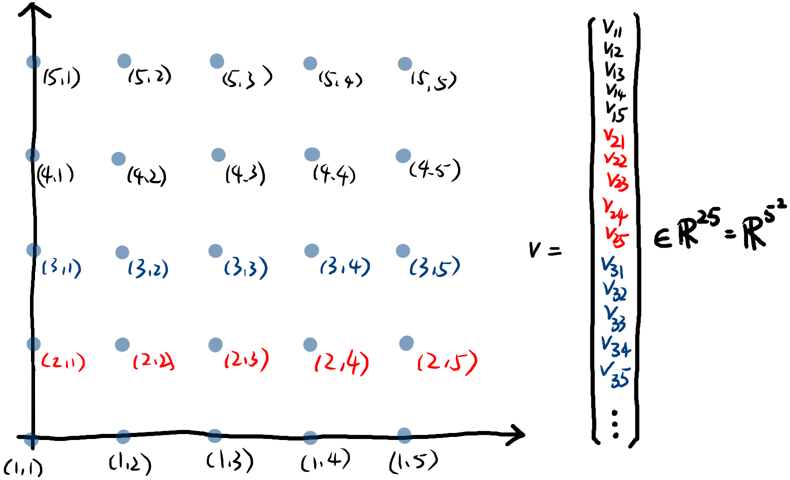
$$\begin{cases} \dfrac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h^2}. + \dfrac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{h^2} = f_{ij} & \forall v_{i,j} \in \Omega \\ v_{i,j} = 0 & \forall v_{i,j} \in \partial\Omega \end{cases}$$

- Re-arrange and combine terms gives

$$\begin{cases} \dfrac{v_{i+1,j} + v_{i,j+1} - 4v_{i,j} + v_{i-1,j} + v_{i,j-1}}{h^2} = f_{ij} & \forall v_{i,j} \in \Omega \\ v_{i,j} = 0 & \forall v_{i,j} \in \partial\Omega \end{cases}$$

A lexicographical ordering of variables

We convert the matrix variable $v_{i,j}$ into a vector following a convention: stacking rows to form a long vector.



Discrete BVP as a linear system of equation

- Now we have $\mathbf{v} = \mathbf{A}\mathbf{f}$ where

$$\mathbf{v} = \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \end{bmatrix}, \mathbf{f} = \begin{bmatrix} f_{11} \\ f_{12} \\ \vdots \end{bmatrix}$$

- Now the question now is, what is $\mathbf{A}$?
  We can construct $\mathbf{A}$ from the linear relation

$$\begin{cases} \dfrac{v_{i+1,j} + v_{i,j+1} - 4v_{i,j} + v_{i-1,j} + v_{i,j-1}}{h^2} = f_{ij} & \forall v_{i,j} \in \Omega \\ v_{i,j} = 0 & \forall v_{i,j} \in \partial\Omega \end{cases}$$

- Now we focus on the example on the grid point $v_{3,3}$.

At v(3,3)

The discrete BVP
$$\begin{cases} \dfrac{v_{i+1,j} + v_{i,j+1} - 4v_{i,j} + v_{i-1,j} + v_{i,j-1}}{h^2} = f_{ij} & \forall v_{i,j} \in \Omega \\ v_{i,j} = 0 & \forall v_{i,j} \in \partial\Omega \end{cases}$$

Now becomes
$$\frac{v_{4,3} + v_{3,4} - 4v_{3,3} + v_{2,3} + v_{3,2}}{h^2} = f_{33}.$$

$$\implies \frac{1}{h^2}\begin{bmatrix} 0\,0\,1\,0\,0 & 0\,1\,-4\,1\,0 & 0\,0\,1\,0\,0 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \\ v_{24} \\ \underline{v_{25}} \\ v_{31} \\ v_{32} \\ v_{33} \\ v_{34} \\ \underline{v_{35}} \\ v_{41} \\ v_{42} \\ v_{43} \\ v_{44} \\ v_{45} \end{bmatrix} = f_{33}.$$

At v(3,3)

The previous long equation corresponds to a segment of a row in $\mathbf{A}$

-4	1				1										v21
1	-4	1				1									v22
	1	-4	1				1								v23
		1	-4	1				1							v24
			1	-4					1						v25
1					-4	1				1					v31
	1				1	-4	1				1				v32
		1				1	-4	1				1			v33
			1				1	-4	1				1		v34
				1				-4	1					1	v35
					1					-4	1				v41
						1				1	-4	1			v42
							1				1	-4	1		v43
								1				1	-4	1	v44
									1				1	-4	v45

The A matrix in the discrete BVP

- For the 25-point example, in general we have $\mathbf{A}_h \mathbf{v}_h = \mathbf{f}_h$, where $\mathbf{A}$ is a block-tridiagonal sparse matrix

$$\mathbf{A} = \frac{1}{h^2} \begin{bmatrix} \mathbf{T} & \mathbf{I} & & & \\ \mathbf{I} & \mathbf{T} & \mathbf{I} & & \\ & \mathbf{I} & \mathbf{T} & \mathbf{I} & \\ & & \mathbf{I} & \mathbf{T} & \mathbf{I} \\ & & & \mathbf{I} & \mathbf{T} \end{bmatrix} \in \mathbb{R}^{25\times 25} = \mathbb{R}^{5^2 \times 5^2},$$

  where $\mathbf{I}$ is a 5-by-5 identity matrix and $\mathbf{T}$ is a tri-diagonal sparse matrix

$$\mathbf{T} = \begin{bmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & 1 & -4 & 1 & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{bmatrix} \in \mathbb{R}^{5\times 5}.$$

- Note that $\mathbf{A}$ has 5^4 number of entries, and most of them are zero. In computation, we have to form $\mathbf{A}$ using sparse construction.

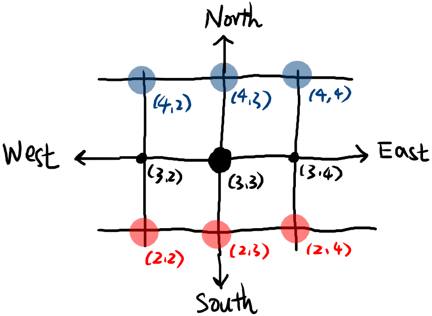
Understanding the row of A

- We now re-look at the row $[0\,0\,1\,0\,0 \quad 0\,1\,-4\,1\,0 \quad 0\,0\,1\,0\,0]$ in the $v_{3,3}$ example.

- Fact: these numbers indicate their relationship with the point in consideration $(v_{3,3})$

	North West	North	North East		West west	West	**The point**	East	East East		South west	South	South east	

- In the $v_{3,3}$ example, we only care about the 4 surrounding points: east (E), west (W), south (S), north (N).

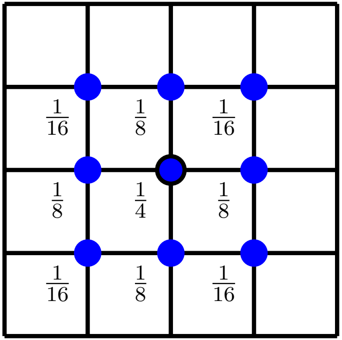


Hence that row of $\mathbf{A}$ has exactly 5 (=4+1) non-zeros.

Two-dimensional restriction operator in multigrid method

- Understanding of the structure of $\mathbf{A}$ is useful for understanding multigrid methods.
- In two-dimensional multigrid, we consider the surrounding 8 points:

$$E, W, S, N, NE, NW, SE, SW.$$



Therefore, the row of the restriction matrix will have (at most) 9 (=8+1) non-zeros.

The two-dimensional restriction operator

- The full weighting operator take the central point with the 8 surrounding points to form the new point through the formula

$$\frac{x_{i,j}}{4} + \frac{x_{i-1,j} + x_{i,j-1} + x_{i+1,j} + x_{i,j+1}}{8} + \frac{x_{i-1,j-1} + x_{i-1,j+1} + x_{i+1,j-1} + x_{i+1,j+1}}{16}$$

- Therefore, based on the previous example of $\begin{bmatrix} 0\,0\,1\,0\,0 & 0\,1\,-4\,1\,0 & 0\,0\,1\,0\,0 \end{bmatrix}$

	North West	North	North East		West west	West	**The point**	East	East East		South west	South	South east	

Now the row of the full weighting operator will be

$$0 \;\frac{1}{16}\;\frac{1}{8}\;\frac{1}{16}\;0 \quad 0\;\frac{1}{8}\;\frac{1}{4}\;\frac{1}{8}\;0 \quad 0\;\frac{1}{16}\;\frac{1}{8}\;\frac{1}{16}\;0$$

Last page - summary

- 2D Poisson equation
- Grid discretization and Taylor series approximation
- Discrete Boundary Value Problem
- The understanding of the row of the matrix $\mathbf{A}$ and its generalization

Not discussed / What's next

- Multigrid method

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