Cauchy-Riemann Condition

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1 Rectangular Form

$$w = f(z) = u(x,y) + jv(x,y)$$

$$\frac{dw}{dz} = \frac{df(z)}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x+\Delta x, y+\Delta y) + jv(x+\Delta x, y+\Delta y)] - [u(x,y) + jv(x,y)]}{\Delta x + j\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x+\Delta x, y+\Delta y) - u(x,y)] + j[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + j\Delta y}$$

Case 1 : Let $\Delta y = 0$

$$= \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) - u(x,y)] + j[v(x+\Delta x, y) - v(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + j \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}$$

Case 2 : Let $\Delta x = 0$

$$= \lim_{\Delta y \to 0} \frac{[u(x+, y+\Delta y) - u(x,y)] + j[v(x, y+\Delta y) - v(x,y)]}{j\Delta y}$$

$$= \lim_{\Delta y \to 0} -j\frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \lim_{\Delta x \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= -j\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

The 2 case should be equal

$$\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = -j\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$$

2 Polar Form

2.1 Method 1

$$\begin{cases} x = r\cos\theta \qquad y = r\sin\theta \\ \\ x^2 + y^2 = r^2 \qquad \tan\theta = \dfrac{y}{x} \end{cases}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial u}{\partial x}r\sin\theta + \frac{\partial u}{\partial y}r\cos\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial v}{\partial x}r\sin\theta + \frac{\partial v}{\partial y}r\cos\theta$$

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta \qquad \dfrac{\partial v}{\partial r} = \dfrac{\partial v}{\partial x}\cos\theta + \dfrac{\partial v}{\partial y}\sin\theta \\ \\ \dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x}r\sin\theta + \dfrac{\partial u}{\partial y}r\cos\theta \qquad \dfrac{\partial v}{\partial \theta} = -\dfrac{\partial v}{\partial x}r\sin\theta + \dfrac{\partial v}{\partial y}r\cos\theta \end{cases}$$

By the rectangular form equation

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$$

$$r\frac{\partial u}{\partial r} = r\frac{\partial u}{\partial x}\cos\theta + r\frac{\partial u}{\partial y}\sin\theta \ \rightarrow \ r\frac{\partial u}{\partial r} = r\frac{\partial v}{\partial y}\cos\theta - r\frac{\partial v}{\partial x}\sin\theta = \frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial x}r\sin\theta + \frac{\partial u}{\partial y}r\cos\theta \ \rightarrow \ \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial y}r\sin\theta - \frac{\partial v}{\partial x}r\cos\theta = -r\frac{\partial v}{\partial r}$$

i.e.

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \\ \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial v}{\partial r} \end{cases}$$

2.2 Method 2

$$\begin{cases} x = r\cos\theta & y = r\sin\theta \\ r = \sqrt{x^2+y^2} & \theta = \tan^{-1}\dfrac{y}{x} \end{cases}$$

Then

$$\begin{cases} \dfrac{\partial r}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r} = \cos\theta & \dfrac{\partial r}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}} = \dfrac{y}{r} = \sin\theta \\ \dfrac{\partial \theta}{\partial x} = \dfrac{-y}{x^2+y^2} = \dfrac{-y}{r^2} = \dfrac{-\sin\theta}{r} & \dfrac{\partial \theta}{\partial y} = \dfrac{x}{x^2+y^2} = \dfrac{x}{r^2} = \dfrac{\cos\theta}{r} \end{cases}$$

i.e.

$$\begin{cases} \dfrac{\partial r}{\partial x} = \cos\theta & \dfrac{\partial r}{\partial y} = \sin\theta \\ \dfrac{\partial \theta}{\partial x} = \dfrac{-\sin\theta}{r} & \dfrac{\partial \theta}{\partial y} = \dfrac{\cos\theta}{r} \end{cases}$$

Using chain rule in Rectangular Form CR-Condition :

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases} \iff \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial \theta}\dfrac{\partial \theta}{\partial x} = \dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial v}{\partial \theta}\dfrac{\partial \theta}{\partial y} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial v}{\partial \theta}\dfrac{\partial \theta}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial \theta}\dfrac{\partial \theta}{\partial y} = \dfrac{\partial u}{\partial y} \end{cases}$$

i.e.

$$\begin{cases} \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial \theta}\dfrac{\partial \theta}{\partial x} = \dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial v}{\partial \theta}\dfrac{\partial \theta}{\partial y} \\ \dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial v}{\partial \theta}\dfrac{\partial \theta}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial \theta}\dfrac{\partial \theta}{\partial y} \end{cases}$$

Plug in

$$\begin{cases} \dfrac{\partial u}{\partial r}\cos\theta - \dfrac{\partial u}{\partial \theta}\dfrac{\sin\theta}{r} = \dfrac{\partial v}{\partial r}\sin\theta + \dfrac{\partial v}{\partial \theta}\dfrac{\cos\theta}{r} \\ \dfrac{\partial v}{\partial r}\cos\theta - \dfrac{\partial v}{\partial \theta}\dfrac{\sin\theta}{r} = \dfrac{\partial u}{\partial r}\sin\theta + \dfrac{\partial u}{\partial \theta}\dfrac{\cos\theta}{r} \end{cases}$$

$$\begin{cases} \left(\dfrac{\partial u}{\partial r} - \dfrac{\partial v}{\partial \theta}\right)\cos\theta = \left(\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} + \dfrac{\partial v}{\partial r}\right)\sin\theta \\ \left(\dfrac{\partial v}{\partial r} - \dfrac{1}{r}\dfrac{\partial u}{\partial \theta}\right)\cos\theta = \left(\dfrac{1}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{\partial u}{\partial r}\right)\sin\theta \end{cases}$$

3 Laplace Equation

The C-R Condition

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases} \quad \text{Rectangular Form} \qquad \begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \\ \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial v}{\partial r} \end{cases} \quad \text{Polar Form}$$

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} \quad \text{Mix Derivative} \;\rightarrow\; \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \;\rightarrow\; \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0$$

$$\iff \nabla^2 u = 0$$

$$\frac{\partial}{\partial x}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$$

$$\iff \nabla^2 v = 0$$

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