# Cauchy's Theorems I

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Augustin-Louis Cauchy
1789 – 1857

**References**

Murray R. Spiegel *Complex Variables with introduction to conformal mapping and its applications*
Dennis G. Zill , P. D. Shanahan *A First Course in Complex Analysis with Applications*
J. H. Mathews , R. W. Howell *Complex Analysis For Mathematics and Engineerng*

## 1 Summary

- **Cauchy Integral Theorem**

Let $f$ be analytic in a **simply connected** domain $D$. If $C$ is a simple closed contour that lies in $D$ , and there is no singular point inside the contour, then

$$\int_C f(z)\,dz = 0$$

- **Cauchy Integral Formula (For simple pole)**

If there is a singular point $z_0$ inside the contour, then

$$\oint \frac{f(z)}{z - z_0} dz = 2\pi j\, f(z_0)$$

- **Generalized Cauchy Integral Formula (For pole with any order)**

$$\oint \frac{f(z)}{(z - z_0)^n} dz = \frac{2\pi j}{(n-1)!}\, f^{(n-1)}(z_0)$$

- **Cauchy Inequality**

$$\left| f^{(n)}(z_0) \right| \le \frac{M \cdot n!}{r^n}$$

- **Gauss Mean Value Theorem**

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + re^{j\theta}\right) d\theta$$

# 2 Related Mathematics Review

## 2.1 Stoke's Theorem

$$\iint_{\Sigma} \nabla \times \bar{F} \cdot d\mathbf{S} = \oint_{\partial\Sigma} \bar{F} \cdot d\mathbf{r}$$

(The proof is skipped)

Consider $\bar{F} = (F_X\,,\, F_Y\,,\, F_Z)$

$$\nabla \times \bar{F} = \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_X & F_Y & F_Z \end{pmatrix}$$

Let $F_Z = 0$

$$\nabla \times \bar{F} = \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_X & F_Y & 0 \end{pmatrix}$$

$d\mathbf{S} = \hat{n} dS$ , for $dS = dxdy$ , $\hat{n} = \hat{z}$ . By $\hat{z} \cdot \hat{z} = 1$ , consider $\hat{z}$ component only for $\nabla \times \bar{F}$

$$\nabla \times \bar{F} = \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_X & F_Y & 0 \end{pmatrix} = \left( \frac{\partial F_Y}{\partial x} - \frac{\partial F_X}{\partial y} \right) \hat{z}$$

i.e.

$$\iint_{\Sigma} \nabla \times \bar{F} \cdot d\mathbf{S} = \iint_{\Sigma} \left( \frac{\partial F_Y}{\partial x} - \frac{\partial F_X}{\partial y} \right) dxdy$$

Consider the RHS (let $F_Z = 0$)

$$\oint_{\partial\Sigma} \bar{F} \cdot d\mathbf{r} = \oint_{\partial\Sigma} (F_X, F_Y, F_Z) \cdot (dx, dy, dz) = \oint_{\partial\Sigma} F_X dx + F_Y dy$$

Thus

$$\iint_{\Sigma} \left( \frac{\partial F_Y}{\partial x} - \frac{\partial F_X}{\partial y} \right) dxdy = \oint_{\partial\Sigma} F_X dx + F_Y dy$$

## 2.2 Cauchy-Reimann Condition

For $f(z) = u(z) + jv(z) = u(x, y) + jv(x, y)$
The complex derivative is

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

If the limit exists.

If the limit exists, limits along real axis should be the same the limit along imaginary axis

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{\partial f(z_0)}{\partial x} = \frac{1}{j}\frac{\partial f(z_0)}{\partial y} = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z_0 + jh) - f(z_0)}{jh}$$

i.e. The Cauchy-Riemann Equation

$$\frac{\partial f(z_0)}{\partial x} = \frac{1}{j}\frac{\partial f(z_0)}{\partial y}$$

Expand

$$\frac{\partial}{\partial x}\left[u(x_0, y_0) + jv(x_0, y_0)\right] = \frac{1}{j}\frac{\partial}{\partial y}\left[u(x_0, y_0) + jv(x_0, y_0)\right]$$

Equalize real part and imaginary part

$$\begin{cases} \frac{\partial}{\partial x}u(x_0, y_0) = \frac{\partial}{\partial y}v(x_0, y_0) \\ \frac{\partial}{\partial x}v(x_0, y_0) = -\frac{\partial}{\partial y}u(x_0, y_0) \end{cases}$$

Or simply as

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$$

i.e. If $f(z)$ is analytic, then it fulfill this condition

## 2.3 ML Inequality

First,

$$\text{if } g(x) \leq f(x)\text{, then } \int_a^b g(x)dx \leq \int_a^b f(x)dx$$

This is true if

$$\sup g(x) \leq \sup f(x) \qquad \inf g(x) \leq \inf f(x)$$

Then for any function $f$ , it is true that

$$-|f| \leq f \leq |f|$$

Then apply the inequality above ,

$$-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$$

Now is the time to show , for bounded $f(z)$ , i.e. $|f(z)| \leq M$ ,

$$\left|\int_c f(z)dz\right| \leq ML$$

Pf.
By using the $\int_a^b f(x)dx \leq \int_a^b |f(x)|dx$

$$\left|\int_c f(z)dz\right| \leq \int_c |f(z)|dz$$

Since $f(z)$ si bounded,

$$\left|\int_c f(z)dz\right| \leq \int_c |f(z)|dz \leq \int_c Mdz = M \underbrace{\int_C dz}_{L} = ML$$

Therefore

$$\left|\int_c f(z)dz\right| \leq ML$$

# 3 The Cauchy Theorem

$$f(z) = u(x,y) + jv(x,y)$$

$$\oint_C f\left(z\right)dz = \oint_C \left[u\left(x,y\right) + jv\left(x,y\right)\right]\left[dx + jdv\right] = \oint_C udx - vdy + j\oint_C vdx + udy$$

## 3.1 The Real Part

Let $F_X = u\left(x,y\right)$ $F_Y = -u\left(x,y\right)$ and consider the real part of the integral

$$\oint_C udx - vdy = \oint F_X dx + F_Y dy$$

By the Stoke' Theorem

$$\oint_C udx + (-v)dy = \iint \left[(-v)_x - u_y\right]dxdy = \iint \left[-v_x - u_y\right]dxdy$$

Since the function is *analytic* in the region $D$ , $\iff$ the function fulfill Cauchy-Reimann Condition $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ Then

$$\iint \left[-v_x - u_y\right]dxdy = \iint \left[u_y - u_y\right]dxdy = 0$$

The real part of $\int_C f(z)dz$ equal zero

### 3.2 The imaginary part

Consider the imaginary part of $\int_C f(z)dz$ ,

$$\oint_C vdx + udy$$

Let $F_X$ be $v(x,y)$ and $F_Y$ be $u(x,y)$ , and apply Stoke's Theorem

$$\oint_C vdx + udy = \iint (u_x - v_y)\, dxdy$$

As the function is analytic, so it fulfill Cauchy-Reimann Condition : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ , thus the integral become zero

Finally,
Let $f$ be analytic in a simply connected domain $D$ , for the simple closed contour $C$ that lies in $D$ , the close contour integral equal zero

$$\int_C f(z)\, dz = 0$$

## 4 The Cauchy Integral Formulas

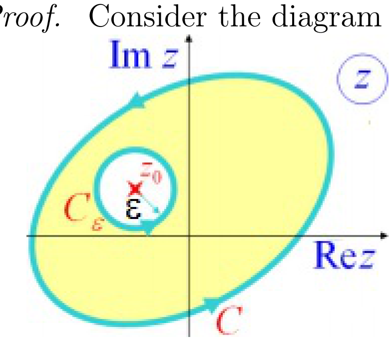
### 4.1 Simple Pole

$$\oint_C \frac{g(z)}{z - z_0} dz = 2\pi j g(z_0)$$

Condition

- $z_0$ inside $C$ , if $z_0$is outside $C$ , then the integral is just zero
- $g(z)$ is analytic in simply connected domain $\iff$ domain of $g(z)$ is simply connect closed region, and $g(z)$ fulfill CR-Condition there

*Proof.* Consider the diagram




By concept of *path deformation* , the integral of $\frac{g(z)}{z-z_0}$ along path $C$ is equivalent to the integral along path $C_\epsilon$ that $\epsilon \to 0$

$$\oint_C \frac{g(z)}{z - z_0} dz = \lim_{\epsilon \to 0} \oint_{C_\epsilon} \frac{g(z)}{z - z_0} dz$$

The small circle inside can be expressed as

$$|z - z_0| = \epsilon e^{j\theta} \quad 0 \le \theta < 2\pi$$

As the radius $\epsilon \to 0^+$ , the circle is thus

$$z - z_0 = \epsilon e^{j\theta} \quad 0 \le \theta < 2\pi$$

Put this back into the integral

$$\oint_C \frac{g(z)}{z-z_0}dz = \lim_{\epsilon\to 0}\oint_{C_\epsilon}\frac{g(z)}{z-z_0}dz = \lim_{\epsilon\to 0}\int_0^{2\pi}\frac{g\left(z_0+\epsilon e^{j\theta}\right)}{\left(z_0+\epsilon e^{j\theta}\right)-z_0}d\left(z_0+\epsilon e^{j\theta}\right)$$

$$= \lim_{\epsilon\to 0}\int_0^{2\pi}\frac{g\left(z_0+\epsilon e^{j\theta}\right)}{\epsilon e^{j\theta}}j\epsilon e^{j\theta}d\theta = j\lim_{\epsilon\to 0}\int_0^{2\pi}g\left(z_0+\epsilon e^{j\theta}\right)d\theta$$

$$= j\int_0^{2\pi}\left[\lim_{\epsilon\to 0}g\left(z_0+\epsilon e^{j\theta}\right)\right]d\theta = j\int_0^{2\pi}g\left(z_0\right)d\theta = jg\left(z_0\right)\int_0^{2\pi}d\theta = 2\pi jg\left(z_0\right)$$

$\therefore$

$$\oint_C \frac{g(z)}{z-z_0}dz = 2\pi jg\left(z_0\right)$$

□

## 4.2 Generalized Cauchy Integral Formula

$$\oint \frac{f\left(z_0\right)}{\left(z-z_0\right)^n}dz = \frac{2\pi j}{(n-1)!}2\pi j\, f^{(n-1)}\left(z_0\right)$$

First, the Cauchy Integral Formula for simple pole is

$$\oint_C \frac{g\left(z\right)}{z-z_0}dz = 2\pi jg\left(z_0\right)$$

Take $\dfrac{d}{dz_0}$ on both side $\left(\text{Not } \dfrac{d}{dz}!\right)$

$$\frac{d}{dz_0}\oint_C\frac{g\left(z\right)}{z-z_0}dz = 2\pi j\frac{d}{dz_0}g\left(z_0\right) \qquad \Rightarrow \oint_C\frac{g\left(z\right)}{\left(z-z_0\right)^2}dz = \frac{2\pi j}{1!}g'\left(z_0\right)$$

Repeat,

$$\frac{d}{dz_0}\oint_C\frac{g\left(z\right)}{\left(z-z_0\right)^2}dz = \frac{2\pi j}{1!}\frac{d}{dz_0}g'\left(z_0\right) \qquad \Rightarrow \oint_C\frac{g\left(z\right)}{\left(z-z_0\right)^3}dz = \frac{2\pi j}{2!}g''\left(z_0\right)$$

$$\frac{d}{dz_0}\oint_C\frac{g\left(z\right)}{\left(z-z_0\right)^3}dz = \frac{2\pi j}{1!}\frac{d}{dz_0}g''\left(z_0\right) \qquad \Rightarrow \oint_C\frac{g\left(z\right)}{\left(z-z_0\right)^4}dz = \frac{2\pi j}{3!}g^{(3)}\left(z_0\right)$$

Thus, the general form of Cauchy Integral Formula is

$$\oint_C\frac{g\left(z\right)}{\left(z-z_0\right)^n}dz = \frac{2\pi j}{(n-1)!}g^{(n-1)}\left(z_0\right)$$

This can be proved using Mathematical Induction.

# 5 Consequences of Cauchy's Integral Formula

## 5.1 Cauchy Inequality

The generalized Cauchy Integral Formula is

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi j}{n!} f^{(n)}(z_0)$$

Take absolute value

$$\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| = \left|\frac{2\pi j}{n!} f^{(n)}(z_0)\right| \qquad \Rightarrow \left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| = \frac{|2\pi|\cdot|j|}{|n!|}\left|f^{(n)}(z_0)\right|$$

Rearrange

$$\left|f^{(n)}(z_0)\right| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \le \frac{n!}{2\pi}\oint_C \left|\frac{f(z)}{(z-z_0)^{n+1}}\right| dz$$

Since

$$z = z_0 + re^{j\theta} \quad \Rightarrow (z-z_0) = re^{j\theta} \quad \Rightarrow \left|(z-z_0)^{n+1}\right| = \left|r^{n+1}\right|\cdot\underbrace{\left|e^{j(n+1)\theta}\right|}_{1} = r^{n+1}$$

And apply ML inequality

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi}\oint_C \left|\frac{f(z)}{r^{n+1}}\right| dz \le \frac{n!}{2\pi\cdot r^{n+1}}\oint_C M dz = \frac{M\cdot n!}{2\pi\cdot r^{n+1}}\underbrace{\oint_C dz}_{2\pi r} = \frac{M\cdot n!}{2\pi\cdot r^{n+1}}\cdot 2\pi r$$

Finally

$$\left|f^{(n)}(z_0)\right| \le \frac{M\cdot n!}{r^n}$$

## 5.2 Gauss Mean Value Theorem

Recall, the Cauchy Integral Formula for simple pole

$$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi j\, f(z_0)$$

Rearrange, and take $z = z_0 + re^{j\theta}$ , $dz = rje^{j\theta}d\theta$

$$f(z_0) = \frac{1}{2\pi j}\oint_C \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi j}\int_0^{2\pi}\frac{f\left(z_0+re^{j\theta}\right)}{re^{j\theta}}\left(rje^{j\theta}d\theta\right)$$

Cancel out common factor

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0+re^{j\theta}\right)d\theta$$

*END*