Cauchy's Theorems II

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References

Murray R. Spiegel *Complex Variables with introduction to conformal mapping and its applications*

1 Summary

- **Louiville Theorem**

  If $f(z)$ is analytic in entire complex plane, and if $f(z)$ is bounded, then $f(z)$ is a constant

- **Fundamental Theorem of Algebra**

  1. $f(z) = \sum_{k=0}^{k=n} a_k z^k = 0$ has at least ONE root, $n \geq 1$, $a_n \neq 0$
  2. $f(z)$ has exactly n roots.

- **The Argument Principle**

  Let Z be number of zeros, P be number of poles of $f(z)$ inside C , then

$$\frac{1}{2\pi j} \oint_c \frac{f'(z)}{f(z)} dz = Z - P$$

2 Consequences of Cauchy's Integral Formula

2.1 Liouville's Theorem

If $f(z)$ is analytic $\forall z$ (entire complex plane) and $f(x)$ is bounded (i.e. $|f(z)| \leq M = \sup f(z)$), then $f(z)$ is a constant.

Pf. By Cauchy Inequality

$$\left|f^{(n)}(a)\right| \leq \frac{M \cdot n!}{r^n}$$

Put $n = 1$, $a = z$

$$\left|\frac{df(z)}{dz}\right| \leq \frac{M}{r}$$

If $r \to \infty$, $\dfrac{df(z)}{dz} = 0$, therefore $f(z) = constant$

2.2 Fundamental Theorem of Algebra

2.2.1 $f(z) = \sum_{k=0}^{k=n} a_k z^k = 0$ has at least ONE root, $n \geq 1$, $a_n \neq 0$.

Assume $f(z) = 0$ has no root, thus $g(z) = \dfrac{1}{f(z)}$ has no pole. Thus it is analytic $\forall z$ (in entire complex plane).

As $f(z) = 0$ has no root, $|g(z)| = \left|\dfrac{1}{f(z)}\right|$ is bounded.

Then, this function fulfill the requirement of Liouville's theorem

By Liouville's theorem, $f(z)$ and thus $g(z)$ must be a constant.

Then there is contraction to $f(z) = 0$ has no root.

Thus, the assumption is false, $f(z)$ has at least one zero, i.e. one root.

2.2.2 $f(z) = \sum_{k=0}^{k=n} a_k z^k = 0$ has exactly n root, $n \geq 1$, $a_n \neq 0$.

$f(z) = 0$ has at least one root, let it be z_0
Consider $f(z) - f(z_0)$

$$f(z) - f(z_0) = \sum_{k=0}^{k=n} a_k z^k - \sum_{k=0}^{k=n} a_k z_0^k = a_0 + a_1 z + a_2 z^2 + ... - \left(a_0 + a_1 z_0 + a_2 z_0^2 + ...\right)$$

$$= a_1 (z - z_0) + a_2 (z - z_0)^2 + ... = \sum_{k=1}^{k=n} a_k (z - z_0)^k = (z - z_0) \sum_{k=1}^{k=n} a_k (z - z_0)^{k-1}$$

$$= (z - z_0) \sum_{k=0}^{k=n-1} a_k (z - z_0)^k = (z - z_0)\, g(z)$$

where $g(z)$ is a polynomial with degree $n - 1$
Repeat this process n times, it shows that $f(z)$ has exactly n root.

2.3 The Argument Principle

Let Z be number of zeros, P be number of poles of $f(z)$ inside C , then

$$\frac{1}{2\pi j} \oint_c \frac{f'(z)}{f(z)} dz = Z - P$$

Consider simple case, there is one zero $\mathcal{Z}$ and one pole $\mathcal{P}$ inside the contour C.
The zero $\mathcal{Z}$ has multiplicity z_1 , $\mathcal{P}$ has multiplicity p_1

So by contour decomposition

$$\frac{1}{2\pi j} \oint_c \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi j} \oint_{c(z)} \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi j} \oint_{c(p)} \frac{f'(z)}{f(z)} dz$$

Consider the first integral $\frac{1}{2\pi j}\oint_{c(z)}\frac{f'(z)}{f(z)}dz$

First, $f(z)$ has a zero $\mathcal{Z}$ of multiplicity z_1 ,so we can factorize the term $(z-\mathcal{Z})^{z_1}$ out

$$f(z)=(z-\mathcal{Z})^{z_1}\,F(z)$$

$F(z)$ is analytic , and thus

$$\frac{f'(z)}{f(z)}=\frac{(z-\mathcal{Z})^{z_1}\,F'(z)+z_1\,(z-\mathcal{Z})^{z_1-1}\,F(z)}{(z-\mathcal{Z})^{z_1}\,F(z)}=\frac{F'(z)}{F(z)}+\frac{z_1}{z-\mathcal{Z}}$$

$\therefore$

$$\frac{1}{2\pi j}\oint_{c(z)}\frac{f'(z)}{f(z)}dz=\frac{1}{2\pi j}\oint_{c(z)}\frac{F'(z)}{F(z)}dz+\frac{1}{2\pi j}\oint_{c(z)}\frac{z_1}{z-\mathcal{Z}}dz$$

Since the $F(z)$ is analytic , so its integral is zero.

And , recall that $\oint(z-a)^m\,dz=2\pi j$ if $m=-1$

$$\frac{1}{2\pi j}\oint_{c(z)}\frac{f'(z)}{f(z)}dz=\frac{1}{2\pi j}\oint_{c(z)}\frac{z_1}{z-\mathcal{Z}}dz=\frac{1}{2\pi j}\cdot z_1\cdot 2\pi j=z_1$$

Then, consider the second integral $\frac{1}{2\pi j}\oint_{c(p)}\frac{f'(z)}{f(z)}dz$

$f(z)$ has a pole P of multiplicity p_1 ,so we can factorize the term $(p-\mathcal{P})^{p_1}$ out

$$f(z)=\frac{G(z)}{(p-\mathcal{P})^{p_1}}$$

$G(z)$ is analytic , and thus

$$\frac{f'(z)}{f(z)}=\frac{\dfrac{G'(z)\,(p-\mathcal{P})^{p_1}-p_1\,(p-\mathcal{P})^{p_1-1}\,G(z)}{(p-\mathcal{P})^{2p_1}}}{\dfrac{G(z)}{(p-\mathcal{P})^{p_1}}}=\frac{G'(z)-p_1\,(p-\mathcal{P})^{-1}\,G(z)}{G(z)}=\frac{G'(z)}{G(z)}-\frac{p_1}{p-\mathcal{P}}$$

$\therefore$

$$\frac{1}{2\pi j}\oint_{c(z)}\frac{f\ z}{f(z)}dz=\frac{1}{2\pi j}\oint_{c(z)}\frac{G\ z}{G(z)}dz-\frac{1}{2\pi j}\oint_{c(z)}\frac{p_1}{p-\mathcal{P}}dz$$

Since the $G(z)$ is analytic , so its integral is zero.
And , recall that $\oint(z-a)^m\,dz=2\pi j$ if $m=-1$

$$\frac{1}{2\pi j}\oint_{c(z)}\frac{f'(z)}{f(z)}dz=-\frac{1}{2\pi j}\oint_{c(z)}\frac{p_1}{p-\mathcal{P}}dz=-\frac{1}{2\pi j}\cdot p_1\cdot 2\pi j=-p_1$$

Combine the 2 integral

$$\frac{1}{2\pi j}\oint_{c}\frac{f'(z)}{f(z)}dz=\frac{1}{2\pi j}\oint_{c(z)}\frac{f'(z)}{f(z)}dz+\frac{1}{2\pi j}\oint_{c(p)}\frac{f'(z)}{f(z)}dz=z_1-p_1$$

Now, if there are finite number of poles and zeros inside the contour,

$$\frac{1}{2\pi j}\oint_{c}\frac{f'(z)}{f(z)}dz=\sum_{\text{All zero}}\frac{1}{2\pi j}\oint_{c(z)}\frac{f'(z)}{f(z)}dz+\sum_{\text{All pole}}\frac{1}{2\pi j}\oint_{c(p)}\frac{f'(z)}{f(z)}dz=\sum z_k+\sum p_k$$

$$\frac{1}{2\pi j}\oint_{c}\frac{f'(z)}{f(z)}dz=N-P$$

END