Methods to find the residue a_{-1}

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Summary

	Condition on f , z_0	Methods
1	By definition	$a_{-1} = \frac{1}{2\pi j} \oint_C f(z)dz$
2	f has Laurent Series	a_{-1} =coefficient of $(z-z_0)^{-1}$
3	Consider $(z-z_0)^m f(z)$ for pole of order m	$a_{-1}^{(m)} = \lim_{z\to z_0} \frac{d}{dz}\left[\frac{(z-z_0)^m}{(m-1)!} f(z)\right]$
4	Let $g(z) = (z-z_0)^m f(z)$ $g(z)$ analytic at z_0	$a_{-1}^{(m)} = \frac{g^{(m)}(z_0)}{(m-1)!}$
5	Consider$(z-z_0)f(z)$ for pole of order 1	$a_{-1} = \lim_{z\to z_0}(z-z_0)f(z)$
6	If f is rational function $f(z) = \frac{p(z)}{q(z)}$	$a_{-1} = \frac{p(z_0)}{q'(z_0)}$

1 Review of related Mathematics

1.1 Cauchy Integral Theorem

$$\oint_C (z-z_0)^m \, dz = \begin{cases} 2\pi j & m=-1 \\ 0 & m \neq -1 \end{cases}$$

1.2 Laurent series

For a complex function $f(z)$ that can be expressed as Laurent Series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = ... + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0)^1 + ...$$

1.3 Residue

Then cosider the contour integration

$$\oint f(z)dz = \oint_C \left[... + a_{-1} (z - z_0)^{-1} + ... \right] dz$$

Consider the following

$$\oint_C A (z - z_0)^m \, dz$$

Let $z = z_0 + \epsilon e^{j\theta}$, then $dz = \epsilon j e^{j\theta} d\theta$, C become 0 to 2π

$$\int_0^{2\pi} A \left(\epsilon e^{j\theta}\right)^m j\epsilon e^{j\theta} d\theta = Aj\epsilon^{m+1} \int_0^{2\pi} e^{j(m+1)\theta} d\theta$$

Since $e^{j\theta} = e^{j(\theta + 2k\pi)}$, so the integral becomes zero for all m that not equal to -1
For $m = -1$, the result is $2\pi j A$

$$\oint_C A (z - z_0)^{-1} dz = A \int_0^{2\pi} \left(\epsilon e^{j\theta}\right)^{-1} \left(\epsilon j e^{j\theta} d\theta\right) = A \int_0^{2\pi} j d\theta = 2\pi j A$$

So,

$$\oint f(z)dz = \oint_C \left[\underbrace{...}_{\text{all zero}} + a_{-1} (z - z_0)^{-1} + \underbrace{...}_{\text{all zero}} \right] dz = \oint_C a_{-1} (z - z_0)^{-1} dz = a_{-1} 2\pi j$$

i.e.

$$\oint f(z)dz = a_{-1} 2\pi j$$

Then a_{-1} is the *residue*

$$a_{-1} = \frac{1}{2\pi j} \oint f(z)dz$$

1.4 Cauchy Integral Formula

For the close contour integration of $f(z)$

$$\oint \frac{f(z)}{z - z_0} dz = 2\pi j \, f(z_0)$$

2 Finding the residue

2.1 By Laurent Series

Since the residue is the a_{-1} in the Laurent series, so the residue can be found by expanding the Laurent series.

2.2 By another method

Sometimes using Laurent series to find the resudue is too time consuming!!!
Now consider another method.

First, consider a $f(z)$ with poles of order m (highest order, no pole with higer order i.e. $a_n = 0$ for $n < -m$)

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \underbrace{... + a_{-m-1}(z-z_0)^{-m-1}}_{\text{All zero}} + a_{-m}(z-z_0)^{-m} + ... + a_0 + a_1(z-z_0) + ...$$

i.e.

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n$$

Recall that the method to change the index is to let $n - m = n'$ and then let $n' = n$

$$\sum_{n=-m}^{\infty} a_n (z-z_0)^n = \sum_{n+m=-m+m}^{\infty} a_{n+m-m}(z-z_0)^{n+m-m} = \sum_{n'=0}^{\infty} a_{n'-m}(z-z_0)^{n'-m} = \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^{n-m}$$

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n = \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^{n-m}$$

Multiply $(z-z_0)^m$ and then take $\frac{d}{dz}$

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n$$

$$\frac{d}{dz}(z-z_0)^m f(z) = \frac{d}{dz}\sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n$$

$$\frac{d}{dz}(z-z_0)^m f(z) = \sum_{n=0}^{\infty} n a_{n-m}(z-z_0)^{n-1}$$

Recall, the term $(z-z_0)^{-1}$ terms does not exist

$$\sum_{n=0}^{\infty} n a_{n-m}(z-z_0)^{n-1} = 0 \cdot \underbrace{a_{-1-m}}_{\text{zero}}(z-z_0)^{n-1} + \sum_{n=1}^{\infty} n a_{n-m}(z-z_0)^{n-1}$$

i.e.

$$\frac{d}{dz}(z-z_0)^m f(z) = \sum_{n=1}^{\infty} n a_{n-m}(z-z_0)^{n-1}$$

Change the index notation again, let $n' = n - 1$, and then denote $n' = n$

$$\sum_{n=1}^{\infty} n a_{n-m}(z-z_0)^{n-1} = \sum_{n-1=1-1}^{\infty} (n-1+1)a_{(n-1+1)-m}(z-z_0)^{n-1} = \sum_{n'=0}^{\infty}(n'+1)a_{(n'+1)-m}(z-z_0)^{n'}$$

Then

$$\frac{d}{dz}\left(z-z_0\right)^m f(z)=\sum_{n=0}^{\infty}(n+1)a_{n-m+1}\left(z-z_0\right)^n$$

Take $\frac{d}{dz}$

$$\frac{d^2}{dz^2}\left(z-z_0\right)^m f(z)=\frac{d}{dz}\sum_{n=0}^{\infty}(n+1)a_{n-m+1}\left(z-z_0\right)^n$$

$$\frac{d^2}{dz^2}\left(z-z_0\right)^m f(z)=\sum_{n=0}^{\infty}(n+1)na_{n-m+1}\left(z-z_0\right)^{n-1}$$

Same, the term $(z-z_0)^{-1}$ also does not exist

$$\sum_{n=0}^{\infty}(n+1)na_{n-m-1}\left(z-z_0\right)^{n-1}=1\cdot 0\cdot a_{-m+1}\left(z-z_0\right)^{-1}+\sum_{n=1}^{\infty}(n+1)na_{n-m+1}\left(z-z_0\right)^{n-1}$$

$$\frac{d^2}{dz^2}\left(z-z_0\right)^m f(z)=\sum_{n=1}^{\infty}(n+1)na_{n-m+1}\left(z-z_0\right)^{n-1}$$

Change index notation again

$$\frac{d^2}{dz^2}\left(z-z_0\right)^m f(z)=\sum_{n=0}^{\infty}(n+2)(n+1)na_{n-m+2}\left(z-z_0\right)^n$$

Keep repeating

$$\frac{d^{m-1}}{dz^{m-1}}\left(z-z_0\right)^m f(z)=\sum_{n=0}^{\infty}(n+2)(n+1)n...(n+m-1)a_{n-m+(m-1)}\left(z-z_0\right)^n$$

$$\frac{d^{m-1}}{dz^{m-1}}\left(z-z_0\right)^m f(z)=\sum_{n=0}^{\infty}(n+2)...(n+m-1)a_{n-1}\left(z-z_0\right)^n$$

Consider the summation

$$\begin{aligned}&\sum_{n=\mathbf{0}}^{\infty}(n{+}2)...(n{+}m{-}1)a_{n-1}\left(z-z_0\right)^n\\&=\left[(n+2)...(n+m-1)a_{n-1}\left(z-z_0\right)^n\right]_{n=0}+\sum_{n=\mathbf{1}}^{\infty}(n{+}2)...(n{+}m{-}1)a_{n-1}\left(z-z_0\right)^n\\&=(m-1)!a_{-1}+\sum_{n=1}^{\infty}(n+2)...(n+m-1)a_{n-1}\left(z-z_0\right)^n\end{aligned}$$

i.e.

$$\frac{d^{m-1}}{dz^{m-1}}\left(z-z_0\right)^m f(z)=(m-1)!a_{-1}+\sum_{n=1}^{\infty}(n+2)...(n+m-1)a_{n-1}\left(z-z_0\right)^n$$

Now take the limt

$$\lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}\left(z-z_0\right)^m f(z)\right]=(m-1)!a_{-1}+\underbrace{\lim_{z\to z_0}\sum_{n=1}^{\infty}(n+2)...(n+m-1)a_{n-1}\left(z-z_0\right)^n}_{ZERO}$$

$$\lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)\right]=(m-1)!a_{-1}$$

Thus, the residue of a order m pole can be found by

$$a_{-1}^{(m)}(z_0)=\frac{1}{(m-1)!}\lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)\right]$$

($a_{-1}^{(m)}$, the m here denote the order of this pole)

That is , the second method to find pole is by using

$$a_{-1}^{(m)}(z_0)=\frac{1}{(m-1)!}\lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)\right]$$

2.3 Further Improvement

For rational function $f(z)$, it can be expressed as a fraction $\dfrac{p(z)}{q(z)}$
Then

$$a_{-1}^{(m)}(z_0)=\frac{1}{(m-1)!}\lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)\right]=\frac{1}{(m-1)!}\lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m \frac{p(z)}{q(z)}\right]$$

Consider $m=1$ (consider the case for simple pole)

$$a_{-1}(z_0)=\lim_{z\to z_0}(z-z_0)f(z)=\lim_{z\to z_0}\left[(z-z_0)\frac{p(z)}{q(z)}\right]$$

Since z_0 is a pole, by definition, pole is a root of the $q(z)$, thus $q(z_0)=0$
Just subtract zero in the $q(z)$

$$a_{-1}(z_0)=\lim_{z\to z_0}\left[(z-z_0)\frac{p(z)}{q(z)-0}\right]=\lim_{z\to z_0}\left[(z-z_0)\frac{p(z)}{q(z)-q(z_0)}\right]$$

$$=\lim_{z\to z_0}\left[\frac{p(z)}{\dfrac{q(z)-q(z_0)}{z-z_0}}\right]=\frac{\lim\limits_{z\to z_0}p(z)}{\left[\lim\limits_{z\to z_0}\dfrac{q(z)-q(z_0)}{z-z_0}\right]}=\frac{p(z_0)}{q'(z_0)}$$

(Provided that those limit exist)
i.e. The residue can be found by

$$a_{-1}(z_0)=\frac{p(z_0)}{q'(z_0)}$$

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