# Application of Residue Calculus in Real Integral

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$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = 2\pi j \sum Res(g\,,\, z_p)$$

$$\int_{-\infty}^{\infty} R(x)dx = 2\pi j \sum Res(R\,,\, z_p)$$

$$\int_{-\infty}^{\infty} R(x)e^{j\omega x}dx = 2\pi j \sum Res\,(f\,,\, z_p)$$

## 1 $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$

Change the integral into complex integral

$$e^{j\theta} = z$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} = \frac{z + z^{-1}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{z + z^{-1}}{2j}$$

$$dz = je^{j\theta}d\theta \qquad d\theta = \frac{dz}{je^{j\theta}} = \frac{dz}{jz}$$

$$\theta \in [0, 2\pi] \to |z| = 1$$

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \oint_{|z|=1} R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2j}\right) \frac{dz}{jz}$$

Then let

$$g(z) = R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2j}\right) \frac{1}{jz}$$

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \oint_{|z|=1} R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2j}\right) \frac{dz}{jz} = \oint_{|z|=1} g(z)dz$$

By residue theorem

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = 2\pi j \sum Res(g\,,\, z_p)$$

## 2 $\int_{-\infty}^{\infty} R(x)dx$

$$\int_{-\infty}^{\infty} R(x)dx = \lim_{R\to\infty} \int_{R}^{R} R(x)dx$$

This integral can be treated as a line segment on the close loop contour complex integral

$$\lim_{R\to\infty} \int_{R}^{R} R(x)dx = \lim_{R\to\infty} \oint_{L} R(z)dz - \lim_{R\to\infty} \int_{C_R} R(z)dz$$

By Jordan's Lemma (Which make use of the ML-inequaity )

$$\lim_{R\to\infty} \int_{C_R} R(z)dz = 0$$

Therefore

$$\int_{-\infty}^{\infty} R(x)dx = \oint_{L} R(z)dz$$

By residue theorem

$$\int_{-\infty}^{\infty} R(x)dx = 2\pi j \sum Res(R\,,\, z_p)$$

*Remark.* According to the nature of the function $R(x)$ ,

$$\int_{-\infty}^{\infty} R(x)dx = 2\pi j \sum_{UHP} Res(R\,,\, z_p) \quad or \quad \int_{-\infty}^{\infty} R(x)dx = 2\pi j \sum_{LHP} Res(R\,,\, z_p)$$

## 3 $\int_{-\infty}^{\infty} R(x)e^{j\omega x}dx$ , $\omega > 0$

$$\int_{-\infty}^{\infty} R(x)e^{j\omega x}dx = \lim_{R\to\infty} \int_{R}^{R} R(x)e^{j\omega x}dx$$

This integral can be treated as a line segment on the close loop contour complex integral

$$\lim_{R\to\infty} \int_{R}^{R} R(x)e^{j\omega x}dx = \lim_{R\to\infty} \oint_{L} f(z)dz - \lim_{R\to\infty} \int_{C_R} f(z)dz$$

Where

$$f(z) = R(z)e^{j\omega z}$$

Then , by Jordan's Lemma again, the second integral is zero

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$$

Thus , by residue theorem

$$\int_{-\infty}^{\infty} R(x)e^{j\omega x}dx = 2\pi j \sum_{UHP} Res\,(f\,,\, z_p)$$

*Remark.* If $\omega$ is negative : $\omega = -\omega$ , then it is Fourier Transform

$$\int_{-\infty}^{\infty} R(x)e^{-j\omega x}dx = \mathcal{F}\{R(x)\}$$

Since $\omega$ is negative, so take the upper half plane ( Otherwise, the integral diverge ! )

$$\int_{-\infty}^{\infty} R(x)e^{-j\omega x}dx = 2\pi j \sum_{UHP} Res\,(f\,,\,z_p)$$

Therefore, the Fourier Transform of a real function $R(x)$ can be computed as

$$\mathcal{F}\{R(x)\} = 2\pi j \sum_{UHP} Res\,(f\,,\,z_p)$$

Further more, since

$$\int_{-\infty}^{\infty} R(x)e^{j\omega x}dx = \int_{-\infty}^{\infty} R(x)\cos\omega x dx + j\int_{-\infty}^{\infty} R(x)\sin\omega x dx$$

Thus

$$\int_{-\infty}^{\infty} R(x)\cos\omega x dx = \mathrm{Re}\left[2\pi j \sum Res\,(f\,,\,z_p)\right]$$

$$\int_{-\infty}^{\infty} R(x)\sin\omega x dx = \mathrm{Im}\left[2\pi j \sum Res\,(f\,,\,z_p)\right]$$

And using property of even and odd function, for even function $R(x)$

$$\int_{-\infty}^{\infty} R(x)e^{j\omega x}dx = \int_{-\infty}^{\infty} R(x)\cos\omega x dx + j\int_{-\infty}^{\infty} R(x)\sin\omega x dx$$

Become

$$2\int_{0}^{\infty} R(x)e^{j\omega x}dx = 2\int_{0}^{\infty} R(x)\cos\omega x dx + 0$$

$$\int_{0}^{\infty} R(x)\cos\omega x dx = \pi j \sum Res\,(f\,,\,z_p)$$

And for odd function $R(x)$

$$\int_{0}^{\infty} R(x)\sin\omega x dx = \pi j \sum Res\,(f\,,\,z_p)$$

# 4 Examples

## 4.1 $\int_0^\infty \frac{\cos ax}{1+x^2}dx$ , $a > 0$

$\frac{\cos ax}{1+x^2}$ is even function, thus

$$\int_{0}^{\infty} \frac{\cos ax}{1+x^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}dx$$

Apply the residue method

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos ax}{1+x^2}dx=\frac{1}{2}\mathrm{Re}\left[2\pi j\sum_{UHP}Res\left(f\ ,\ z_p\right)\right]$$

Where

$$f(z)=\frac{e^{jaz}}{1+z^2}\ \text{with pole}\pm j\quad,-j\,\text{rejected (in lower half plane)}$$

i.e.

$$\int_{0}^{\infty}\frac{\cos ax}{1+x^2}dx=\pi\mathrm{Re}\left[j\left[\left(\frac{e^{jaz}}{z+j}\right)_{z=j}\right]\right]=\frac{\pi}{2}e^{-a}$$

## 4.2 $\int_0^\infty \frac{dx}{x^4+a^4}$

$\frac{1}{x^4+a^4}$ is even function, since

$$\int_0^\infty\frac{dx}{x^4+a^4}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^4+a^4}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dz}{z^4+a^4}=2\pi j\sum_{UHP}\mathrm{Res}(f,z_p)$$

There are 4 poles : $a_k=e^{\frac{2k+1}{4}\pi}_{k=0,1,2,3}$ , only first 2 poles in upper half plane

$$\int_0^\infty\frac{dx}{x^4+a^4}=\frac{1}{2}\cdot 2\pi j\left[\mathrm{Res}(f,a_0)+\mathrm{Res}(f,a_1)\right]=\pi j\left[\mathrm{Res}(f,a_0)+\mathrm{Res}(f,a_1)\right]$$

$$=\pi j\left[\frac{a_0+a_1}{-4a^4}\right]=-\frac{\pi j}{4a^4}\left[ae^{j\frac{\pi}{4}}+ae^{j\frac{3\pi}{4}}\right]=\frac{\pi}{2\sqrt{2}a^3}$$

## 4.3 $\int_{-\infty}^\infty \frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx \quad a,b>0$

$f(z)=\frac{z^2}{\left(z^2+a^2\right)\left(z^2+b^2\right)}$ , poles : $\pm ja$ , $\pm jb$ , only $ja$ , $jb$ on upper half plane

$$\int_{-\infty}^{\infty}\frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx=2\pi j\left[\mathrm{Res}\left(f,ja\right)+\mathrm{Res}\left(f,jb\right)\right]$$

$$=2\pi j\left[\frac{z^2}{\left(x+ja\right)\left(x^2+b^2\right)}|_{z=ja}+\frac{z^2}{\left(x^2+a^2\right)\left(x+jb\right)}|_{z=jb}\right]=2\pi j\left[\frac{-a^2}{2ja\left(b^2-a^2\right)}+\frac{-b^2}{\left(a^2-b^2\right)2jb}\right]=\frac{\pi}{a+b}$$

## 4.4 $\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta},\ a>|b|$

$$\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}=\oint_{|z|=1}\frac{1}{a+b\frac{z-z^{-1}}{2j}}\frac{dz}{jz}=2\oint_{|z|=1}\frac{dz}{bz^2+2ajz-b}$$

Find the poles

$$p_{1,2} = \frac{-2aj \pm \sqrt{(2aj)^2 + 4b^2}}{2b} = j\frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

Consider the pole's location to see if they are inside or outside of the contour

$$|p_{1,2}| = \left|\frac{-a \pm \sqrt{a^2 - b^2}}{b}\right| = \left|\frac{\pm\sqrt{a^2 - b^2} - a}{b} \cdot \frac{\pm\sqrt{a^2 - b^2} + a}{\pm\sqrt{a^2 - b^2} + a}\right| = \left|\frac{\pm(a^2 - b^2) - a^2}{\pm\left(\sqrt{a^2 - b^2} + a\right)b}\right|$$

Only $p_1$ is inside the contour

$$|p_1| = \left|\frac{(a^2 - b^2) - a^2}{\left(\sqrt{a^2 - b^2} + a\right)b}\right| = \left|\frac{b}{\sqrt{a^2 - b^2} + a}\right| < \left|\frac{b}{a}\right| < 1 \qquad (\text{since } a > |b|)$$

Thus

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = 2\pi j \text{Res}\left(\frac{1}{bz^2 + 2ajz - b}, p_1\right) = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

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