Important Contour Integrals in Complex Variable

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An important integral 1

$$\oint_{|z|=r} \frac{dz}{z}$$

In polar form , $z = re^{j\theta}$, $dz = rje^{j\theta}d\theta$

$$= \int_0^{2\pi} \frac{rje^{j\theta}d\theta}{re^{j\theta}} = \int_0^{2\pi} jd\theta = 2\pi j$$

An important integral 2

$$\oint_\gamma \frac{dz}{(z-z_0)^n}$$

Path γ : circle of $z_0 + re^{j\theta}$

$$\oint_\gamma \frac{jre^{j\theta}d\theta}{(z_0 + re^{j\theta} - z_0)^n} = \oint \frac{jre^{j\theta}}{r^n e^{jn\theta}}d\theta = jr^{1-n}\oint e^{j(1-n)\theta}d\theta = \begin{cases} j\int_0^{2\pi} d\theta = 2\pi j & n = 1 \\ r^{1-n}j\int_0^{2\pi} e^{j(1-n)\theta} = 0 & n \neq 1 \end{cases}$$

$\therefore$

$$\oint_\gamma \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi j & n = 1 \\ 0 & n \neq 1 \end{cases}$$

Cauchy Integral Formula

For $f(z)$ is analytic in a region D , then $\forall$ **simple** closed curve $C \in D$,

$$\oint_C f(z)dz = 0$$

Result from Cauchy Integral Formula

- $\int_{z_1}^{z_2} f(z)dz$ is path independent / Path deformation is possible
- $\exists F(z)$ s.t. $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$, i.e. $F'(z) = f(z)$
- Since path deformation is possible, the integral path deformed into circles surrounding singular points **which is inside the contour**.
- If the singular points is first order pole, then apply $\oint_\gamma \frac{dz}{(z-z_0)^1} = 2\pi j$ (This equation is true for any simple pole !)

Cauchy Integral Theorem

$$\oint \frac{f(z)}{z - z_0} dz = 2\pi j \cdot f(z_0)$$

$$\oint \frac{f(z)}{(z - z_0)^2} dz = 2\pi j \cdot f'(z_0) \qquad \oint \frac{f(z)}{(z - z_0)^3} dz = \frac{2\pi j}{2!} f''(z_0)$$

$$\oint \frac{f(z)}{(z - z_0)^n} dz = \frac{2\pi j}{(n-1)!} f^{(n-1)}(z_0)$$

$$-END-$$