PDE, BVP and Fourier Series

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Reference Murray R. Spiegel *Fourier Analysis with application to Boundary Value Problems*

1 Reveiw of related Mathematics

1.1 Standard Solution of ODEs

$$\frac{dy}{dx} + \alpha y = 0 \qquad \Rightarrow \qquad y = Ae^{-\alpha x}$$

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \qquad \Rightarrow \qquad y = A\cos\lambda x + B\sin\lambda x$$

$$\frac{d^2y}{dx^2} - \lambda^2 y = 0 \qquad \Rightarrow \qquad y = A\cosh\lambda x + B\sinh\lambda x$$

Short hand notation

$$\frac{dy}{dx} = y' \qquad \frac{d^2y}{dx^2} = y" \qquad \frac{d^ny}{dx^n} = y^{(n)}$$

1.2 Fourier Series

A periodic function with period $2T$ can be represented by sin/cos series - The Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

Where

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos nx dx \qquad b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin nx dx$$

Since the function is periodic with period T , the range of integration can be changed

$$\int_{-T}^{T} = \int_{t_0-T}^{t_0+T} = \int_{0}^{2T}$$

If we plug in the a_n and b_n , the Fourier Series can also be expressed as

$$f(x) = \frac{1}{2T}\left[\int_{-T}^{T} f(x)dx\right] + \frac{1}{T}\sum_{n=1}^{\infty}\left[\int_{-T}^{T} f(x)\cos nx dx\right]\cos nx + \left[\int_{-T}^{T} f(x)\sin nx dx\right]\sin nx$$

Aperiodic function that defined in finite interval $[0, L]$ can be made into periodic by repeating the function

$$f(x) = f(x + L)$$

2 Standard Steps on solving PDETerms

PDE =	Partial Differential Equation
BVP =	Boundary Value Problem
Fourier Series =	Sine/Cosine/e-Series

- Separation of Variables
- Sign of Separation Constant
- Solve Separated ODEs
- Combine Solution and Apply Boundary Condition
- Solve the coefficient by Fourier Series

3 Examples 1

$$u(x,y) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \begin{array}{c} u(0,y) = 0 \\ u(x,0) = 0 \\ u(1,y) = 0 \\ u(x,1) = a \\ |u(x,y)| < M \end{array}$$

1. Separation of Variables

Let $u(x,y) = X(x)Y(y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Longrightarrow \frac{\partial^2 X(x)Y(y)}{\partial x^2} + \frac{\partial^2 X(x)Y(y)}{\partial y^2} = 0$$

Rearrange by divide both side with $X(x)Y(y)$

$$Y(y)\frac{\partial^2 X(x)}{\partial x^2} + X(x)\frac{\partial^2 Y(y)}{\partial y^2} = 0$$

$$\frac{\partial^2 X(x)}{X(x)\partial x^2} + \frac{\partial^2 Y(y)}{Y(y)\partial y^2} = 0 \qquad \Longleftrightarrow \qquad \frac{\partial^2 X(x)}{X(x)\partial x^2} = -\frac{\partial^2 Y(y)}{Y(y)\partial y^2}$$

2. Separation Constant $-\lambda^2$

Since the RHS is y-dependent (i.e. When y is changing, the othere side no change , there is no y in the x-side), and LHS and x-dependent (i.e. The other side of the equation is still constant when x change). So , the equation should equal to a constant

To make use of the ODEs $Y" + \lambda^2 Y = 0$, set the separation constant as $-\lambda$

$$\frac{\partial^2 X(x)}{X(x)\partial x^2} = -\frac{\partial^2 Y(y)}{Y(y)\partial y^2} = -\lambda^2$$

3. Solve separated ODE

$$\frac{\partial^2 X(x)}{X(x)\partial x^2} = -\frac{\partial^2 Y(y)}{Y(y)\partial y^2} = -\lambda^2 \Longrightarrow \begin{cases} \dfrac{\partial^2 X(x)}{X(x)\partial x^2} = -\lambda^2 \\ \\ \dfrac{\partial^2 Y(y)}{Y(y)\partial y^2} = \lambda^2 \end{cases}$$

$$\Longrightarrow \begin{cases} \dfrac{\partial^2 X(x)}{\partial x^2} + \lambda^2 X(x) = 0 \\ \\ \dfrac{\partial^2 Y(y)}{\partial y^2} - \lambda^2 Y(y) = 0 \end{cases} \Longrightarrow \begin{cases} X(x) = A\cos\lambda x + B\sin\lambda x \\ \\ Y(y) = C\cosh\lambda y + D\sinh\lambda y \end{cases}$$

4. Combine Solution and solve BVP

$$u(x,y) = X(x)Y(y) = (A\cos\lambda x + B\sin\lambda x)(C\cosh\lambda y + D\sinh\lambda y)$$

The Boundary Conditions are

$$\begin{aligned} u(0,y) &= 0 \\ u(x,0) &= 0 \\ u(1,y) &= 0 \\ u(x,1) &= a \\ |u(x,y)| &< M \end{aligned}$$

$$u(0,y) = 0 = A(C\cosh\lambda y + D\sinh\lambda y) \quad \Rightarrow A = 0 \quad \text{Update } u(x,y) = \sin\lambda x\,(C'\cosh\lambda y + D'\sinh\lambda y)$$

$$u(x,0) = 0 = \sin\lambda x C' \quad \Rightarrow C' = 0 \quad \text{Update } u(x,y) = E\sin\lambda x\sinh\lambda y$$

$$u(1,y) = 0 = E\sin\lambda\sinh\lambda y \quad \Rightarrow \begin{cases} E = 0 & \text{No, if E=0, all = 0} \\ \text{sinh } \lambda\text{y=0} & \text{No, if } \sinh\lambda y = 0 \text{ , all =0} \\ \sin\lambda = 0 & \text{Only choice} \end{cases}$$

$$\sin\lambda = 0 \quad \Longleftrightarrow \quad \lambda = m\pi \qquad E = E_m$$

$$\text{Update } u_m(x,y) = E_m \sin m\pi x \sinh\lambda y$$

By principle of superposition, the solution is summation of all possible solution

$$U(x,y) = \sum_{m=1}^{\infty} u_m(x,y) = \sum_{m=1}^{\infty} E_m \sin m\pi x \sinh\lambda y$$

5. Apply Fourier Series to solve for Fourier Coefficient

Now apply the last condition to solve for the last unknown, E_m

$$u(x,1) = a = \sum_{m=1}^{\infty} E_m \sin m\pi x \sinh\lambda$$

Rearrange, to make it into the form of Fourier Series

$$u(x,1) = a = \sum_{m=1}^{\infty} (E_m \sinh\lambda)\sin m\pi x$$

Compare it to the standard Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

Then $f(x) = a$, $a_n = 0$, $b_n = E_m \sinh\lambda$

And using the relation of $f(x)$ and the Fourier coefficient ,

$$b_n = \frac{1}{T}\int_0^{2T} f(x)\sin nx dx$$

So

$$E_m \sinh\lambda = \frac{1}{1/2}\int_0^1 a\sin m\pi x dx = \frac{-2a}{m\pi}\cos m\pi x|_0^1 = \frac{2a\left(1-\cos m\pi\right)}{m\pi} = \frac{2a\left(1-(-1)^m\right)}{m\pi}$$

Thus the last unknown is also solved as

$$E_m = \frac{2a\left(1-(-1)^m\right)}{m\pi\sinh\lambda}$$

Therefore, the solution of this PDE under the BVP is

$$U(x,y) = \sum_{m=1}^{\infty}\frac{2a\left(1-(-1)^m\right)}{m\pi\sinh\lambda}\sin m\pi x\sinh\lambda y = \frac{2a}{\pi\sinh\lambda}\sum_{m=1}^{\infty}\frac{\left(1-(-1)^m\right)}{m}\sin m\pi x\sinh\lambda y$$

4 Example 2 String Wave Equation

$$y(x,t) \qquad \frac{\partial^2 y}{\partial t^2} = \alpha^2\frac{\partial^2 y}{\partial x^2}$$

$y(0,t) = 0$: String is fix in both end
$y(L,t) = 0$: String is fix in both end
$y(x,0) = f(x)$: Initial shape
$y_t(x,0) = 0$: Zero Initial speed

$Y(x,t) = X(x)T(t) = XT$

$$XT" = \alpha^2 TX" \qquad \Longrightarrow \frac{X"}{X} = \frac{1}{\alpha^2}\frac{T"}{T} = -\lambda^2$$

$$X" + \lambda X = 0 \qquad X = A\cos\lambda x + B\sin\lambda x$$

$$T" + \lambda^2 a^2 T = 0 \qquad T = C\cos\lambda\alpha x + D\sin\lambda\alpha x$$

$$y(x,t) = XT = \left(A\cos\lambda x + B\sin\lambda x\right)\left(C\cos\lambda\alpha t + D\sin\lambda\alpha t\right)$$

$$y(0,t) = 0 = A\left(C\cos\lambda\alpha t + D\sin\lambda\alpha t\right) \Rightarrow A = 0 \quad y(x,t) = \sin\lambda x\left(C'\cos\lambda\alpha t + D'\sin\lambda\alpha t\right)$$

$$y(L,t) = 0 = \sin\lambda L\left(C'\cos\lambda\alpha t + D'\sin\lambda\alpha t\right) \Rightarrow \lambda L = m\pi \Rightarrow \lambda = \frac{m\pi}{L}$$

$$y(x,t) = \sin\frac{m\pi}{L}x\left(C'\cos\frac{m\pi}{L}\alpha t + D'\sin\frac{m\pi}{L}\alpha t\right)$$

Since $y(x,0) = f(x)$ is for using Fourier Series, use this condition later

Notice that $y_t(x,0)$ is derivative of $y(t)$

$$y_t(x,t) = \sin\frac{m\pi}{L}x\left(-C'\frac{m\pi\alpha}{L}\sin\frac{m\pi}{L}\alpha t + D'\frac{m\pi\alpha}{L}\cos\frac{m\pi}{L}\alpha t\right)$$

$$y_t(x,0)=0=\sin\frac{m\pi}{L}x\left(D'\frac{m\pi\alpha}{L}\right)\quad\Rightarrow D'=0$$

$$y(x,t)=\sin\frac{m\pi}{L}x\cdot C'\cos\frac{m\pi}{L}\alpha t$$

In terms of superposition

$$y_m(x,t)=E_m\sin\frac{m\pi}{L}x\cos\frac{m\pi}{L}\alpha t\qquad Y(x,t)=\sum_{m=1}^{\infty}E_m\sin\frac{m\pi}{L}x\cos\frac{m\pi}{L}\alpha t$$

$$y(x,0)=f(x)=\sum_{m=1}^{\infty}E_m\sin\frac{m\pi}{L}x$$

Now use Fourier Relation with period = length of string = L

$$E_m=\frac{1}{L/2}\int_0^L f(x)\sin\frac{m\pi}{L}xdx=\frac{2}{L}\int_0^L f(x)\sin\frac{m\pi}{L}xdx$$

Thus, the solution of the PDE is

$$Y(x,t)=\frac{2}{L}\sum_{m=1}^{\infty}\left(\int_0^L f(x)\sin\frac{m\pi}{L}xdx\right)\sin\frac{m\pi}{L}x\cos\frac{m\pi}{L}\alpha t$$

5 Example 3 Square Shape Drum Membrane

$$z(x,y,t)\qquad \frac{\partial z}{\partial t}=\alpha^2\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)$$

$$\begin{gathered}|z(x,y,t)|<M\\ z(0,y,t)=0\ \text{Membrane is fixed in end points}\\ z(x,0,t)=0\ \text{Membrane is fixed in end points}\\ z(L,y,t)=0\ \text{Membrane is fixed in end points}\\ z(x,W,t)=0:\ \text{Membrane is fixed in end points}\\ z(x,y,0)=f(x,y):\ \text{Initial Shape}\\ z_t(x,y,0)=0:\ \text{Zero inital vibration}\end{gathered}$$

Where $L=x_{max}$ = length of the membrane, $W=y_{max}$ = width of the membrane

$$z=XYT\Longrightarrow\frac{T"}{\alpha^2T}=\frac{X"}{X}+\frac{Y"}{Y}=-\lambda^2\Rightarrow\begin{cases}\dfrac{T"}{\alpha^2T}=-\lambda^2\\ \dfrac{X"}{X}=-\dfrac{Y"}{Y}-\lambda^2=-\mu^2\Rightarrow\begin{cases}\dfrac{X"}{X}=-\mu^2\\ \dfrac{Y"}{Y}=-\lambda^2+\mu^2\end{cases}\end{cases}$$

$$\begin{cases}X"+\mu^2X=0\\ Y"+(\lambda^2-\mu^2)\,Y=0\\ T"+\alpha^2\lambda^2T=0\end{cases}\Rightarrow\begin{cases}X=A\cos\mu x+B\sin\mu x\\ Y=C\cos\sqrt{\lambda^2-\mu^2}y+D\sin\sqrt{\lambda^2-\mu^2}y\\ T=E\cos\lambda\alpha t+F\sin\lambda\alpha t\end{cases}$$

$$z(x,y,t)=(A\cos\mu x+B\sin\mu x)\left(C\cos\sqrt{\lambda^2-\mu^2}y+D\sin\sqrt{\lambda^2-\mu^2}y\right)(E\cos\lambda t+F\sin\lambda\alpha t)$$

Boundary Condition

$$z(0,y,t)=0\ \Rightarrow A=0\qquad z(x,0,t)=0\ \Rightarrow C=0$$

$$z = B \sin \mu x \left(D \sin \sqrt{\lambda^2 - \mu^2} y \right) (E \cos \lambda t + F \sin \lambda t)$$

To use the condition z_t , take time derivative

$$z_t = z = B \sin \mu x \left(D \sin \sqrt{\lambda^2 - \mu^2} y \right) (-E\lambda \sin \lambda t + F\lambda \cos \lambda \alpha t)$$

$$z_t(x, y, 0) = 0 \quad \Longrightarrow F = 0$$

$$z = B \sin \mu x \left(D \sin \sqrt{\lambda^2 - \mu^2} y \right) E \cos \lambda t = K \sin \mu x \sin \sqrt{\lambda^2 - \mu^2} y \cos \lambda \alpha t$$

$$z(L, y, t) = 0 \Longrightarrow \sin \mu L = 0 \Rightarrow \mu = \frac{m\pi}{L}$$

$$z(x, W, t) = 0 \Longrightarrow \sin \sqrt{\lambda^2 - \mu^2} W = 0 \Rightarrow \sqrt{\lambda^2 - \mu^2} W = n\pi$$

$$\sqrt{\lambda^2 - \mu^2} = \frac{n\pi}{W} \quad \Rightarrow \lambda = \pi \sqrt{\frac{n^2}{W^2} + \frac{m^2}{L^2}} = \underset{\text{if W=L}}{\frac{\pi}{L}\sqrt{n^2 + m^2}} = \underset{\text{if L=1}}{\pi \sqrt{n^2 + m^2}}$$

If $L = 1$, then

$$z = B \sin m\pi x \left(D \sin \sqrt{\lambda^2 - \mu^2} y \right) E \cos \lambda \alpha t = K_{mn} \sin m\pi x \sin n\pi y \cos \sqrt{n^2 + m^2} \pi \alpha t$$

By superposition

$$Z(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} K_{mn} \sin m\pi x \sin n\pi y \cos \sqrt{n^2 + m^2} \pi \alpha t$$

Apply $z(x, y, 0) = f(x, y)$

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} K_{mn} \sin m\pi x \sin n\pi y \qquad \text{Double Fourier Series}$$

$$K_{mn} = \frac{1}{W/2} \frac{1}{L/2} \int_0^W \int_0^L f(x, y) \sin m\pi x \sin n\pi y \, dx dx$$

For $W = L = 1$

$$K_{mn} = 4 \int_0^1 \int_0^1 f(x, y) \sin m\pi x \sin n\pi y \, dx dx$$

Thus the solution of the PDE is

$$Z(x, y, t) = 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\int_0^1 \int_0^1 f(x, y) \sin m\pi x \sin n\pi y \, dx dx \right) \sin m\pi x \sin n\pi y \cos \sqrt{n^2 + m^2} \pi \alpha t$$

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