The Airy's Equation

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Reference Dr. Abul Hasan Siddiqi's Lecture Notes on Differential Equation, Chapter 6.

Introduction

Airy's Equation $\frac{d^2y}{dx^2} + xy = 0$

The idea to solve this equation is : first assume solution exists , but the solution is not in form of elementary function, rather, it is in form of infinite power series.

Then , since the power series is in the form $\sum c_k (x-a)^k$, where c_k is the coefficient, thus by solving the coefficient c_k, the solution can be found.

Step by step solution

Assume $\sum c_k (x-a)^k$ is the solution of the Airy's eqaution $\frac{d^2y}{dx^2} + xy = 0$, that is,

$$y = \sum_{k=0}^{\infty} c_k (x-a)^k$$

Since for the Airy's equation, 0 is an *ordinary point* , so the solution become simply

$$y = \sum_{k=0}^{\infty} c_k x^k$$

And therefore

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} c_k k x^{k-1} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} c_k k(k-1) x^{k-2}$$

Replace $\frac{d^2y}{dx^2}$ and y in the Airy's eqaution by the infinite series solution

$$\sum_{k=0}^{\infty} c_k k(k-1) x^{k-2} + x \sum_{k=0}^{\infty} c_k x^k = 0$$

Thus

$$\sum_{k=2}^{\infty} c_k k(k-1)x^{k-2} + \sum_{k=0}^{\infty} c_k x^{k+1} = 0$$

$$2c_2 + \sum_{k=3}^{\infty} c_k k(k-1)x^{k-2} + \sum_{k=0}^{\infty} c_k x^{k+1} = 0$$

It will be better to combine the two summation sign

For all the k in the first term , perform the following

$$k = k + 2 - 2$$

For all the k in the second term, perform the following

$$k = k + 1 - 1$$

$$2c_2 + \sum_{k+2-2=3}^{\infty} c_{k+2-2}(k+2-2)(k+2-2-1)x^{k+2-2-2} + \sum_{k+1-1=0}^{\infty} c_{k+1-1}x^{k+1-1+1} = 0$$

And rename $k-2$ as n for the first term, and rename $k+1$ as n for the second term

$$2c_2 + \sum_{n+2=3}^{\infty} c_{n+2}(n+2)(n+1)x^n + \sum_{n-1=0}^{\infty} c_{n-1}x^n = 0$$

$$2c_2 + \sum_{n=1}^{\infty} c_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} c_{n-1}x^n = 0$$

Now both summation sign has same starting index $n = 1$ and same power of x

$$2c_2 + \sum_{n=1}^{\infty} \left[c_{n+2}(n+2)(n+1) + c_{n-1}\right] x^n = 0$$

Note : Although the index of summation sign has changed, but the result remains unchange, e.g. $\sum_{k=0}^{5} x^k = \sum_{m=1}^{6} x^{m-1}$

Therefore

$$\frac{d^2y}{dx^2} + xy = 0 \qquad \text{becomes} \qquad 2c_2 + \sum_{n=1}^{\infty} \left[c_{n+2}(n+2)(n+1) + c_{n-1}\right] x^n = 0$$

And now consider

$$2 \underbrace{c_2}_{ZERO} + \underbrace{\sum_{n=1}^{\infty} \left[c_{n+2}(n+2)(n+1) + c_{n-1}\right] x^n}_{ZERO} = \underbrace{0}_{ZERO}$$

Since the equation equal to zero, all term has to be equal to zero

$$\begin{cases} 2c_2 = 0 \\ c_{n+2}(n+2)(n+1) + c_{n-1} = 0 \quad \text{for all } n \end{cases}$$

Thus

$$c_2 = 0$$

And

$$c_{n+2}(n+2)(n+1) + c_{n-1} = 0$$

Which implies

$$c_{n+2} = \frac{-1}{(n+1)(n+2)} c_{n-1}$$

Then we can solve for a c_n for $n = 2, 3, 4, ...$ in terms of c_1 and c_0

$$\begin{cases} c_2 = 0 \\ \text{Put n=1 : } c_3 = \frac{-1}{2 \cdot 3} c_0 \\ \text{Put n=2 : } c_4 = \frac{-1}{3 \cdot 4} c_1 \\ \text{Put n=3 :} c_5 = \frac{-1}{4 \cdot 5} c_2 = 0 \\ \text{Put n=4 :} c_6 = \frac{-1}{5 \cdot 6} c_3 = \frac{+1}{2 \cdot 3 \cdot 5 \cdot 6} c_0 \end{cases} \quad \begin{cases} \text{Put n=5 :} c_7 = \frac{-1}{6 \cdot 7} c_4 = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} c_1 \\ \text{Put n=6 : } c_8 = \frac{-1}{7 \cdot 8} c_5 = 0 \\ \text{Put n=7 : } c_9 = \frac{-1}{8 \cdot 9} c_6 = \frac{-1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} c_0 \\ \text{Put n=8 :} c_{10} = \frac{-1}{9 \cdot 10} c_7 = \frac{-1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} c_1 \\ \text{Put n=9 :} c_{11} = \frac{-1}{11 \cdot 12} c_8 = 0 \end{cases}$$

Therefore the solution of the Airy's eqaution is

$$y = \sum_{k=0}^{\infty} c_k x^k$$

$$y = c_0 + c_1 + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_{10} x^{10} + ...$$

Group terms with c_0 c_1

$$\begin{aligned} y = \quad & c_0 \left(1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + ... + \frac{(-1)^k}{2 \cdot 3 ... (3k-1) 3k} x^{3k} + ...\right) \\ & + c_1 \left(x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 7 \cdot 7} x^7 - \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + ... + \frac{(-1)^k}{3 \cdot 4 ... 3k(3k+1)} x^{3k+1} + ...\right) \end{aligned}$$

Therefore

$$y = c_0 \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2 \cdot 3 ... (3k-1) 3k} x^{3k}\right) + c_1 \left(x + \sum_{k=1}^{\infty} \frac{(-1)^k}{3 \cdot 4 ... 3k(3k+1)} x^{3k+1}\right)$$

is the general solution of the Airy's Equation