Zero Order Bessel Function

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The solution of the zeroth order Bessel's Equation

$$t^2 \frac{d^2 y(t)}{dt^2} + t \frac{d}{dt} y(t) + t^2 y = 0$$

Is

$$y(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} (k!)^2} \cdot t^{2k}$$

1 Review of related mathematics

1.1 Binomial Expansion

$$(a+b)^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r$$

A special case ,

$$(1+x)^n = \sum_{r=0}^{n} C_r^n x^r$$

The binomial coefficient C_r^n

$$C_r^n = \frac{n!}{(n-r)!r!} \qquad n, r \in \mathbb{Z}^+ \quad n > r$$

For $n \in \mathbb{R}$ and $r > 0$

$$C_r^n = \frac{n \cdot (n-1) \cdot (n-2) \ldots (n-r+1)}{r!}$$

1.2 Laplace Transfrom

$$\mathcal{L}\left\{x(t)\right\} = \int_0^\infty x(t)e^{-st}dt = X(s)$$

Property 1

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = sX(s) - x(0)$$

Pf.

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = \int_0^\infty \left(\frac{d}{dt}x(t)\right)e^{-st}dt = \int_0^\infty e^{-st}dx(t) = e^{-st}x(t)|_0^\infty - \int_0^\infty x(t)de^{-st}$$

$$= -x(0) + s\underbrace{\int_0^\infty x(t)e^{-st}dt}_{X(s)} = sX(s) - x(0)$$

And thus

$$\mathcal{L}\left\{\frac{d^2x(t)}{dt^2}\right\} = \mathcal{L}\left\{\frac{d}{dt}\underbrace{\left(\frac{dx(t)}{dt}\right)}_{f(t)}\right\} = s\,\underbrace{\mathcal{L}\left\{\frac{dx(t)}{dt}\right\}}_{F(s)} - \frac{dx(t)}{dt}|_{x=0} = s^2X(s) - sx(0) - x'(0)$$

Property 2

$$\mathcal{L}\left\{tx(t)\right\} = -\frac{d}{ds}X(s)$$

Pf.

$$-\frac{d}{ds}X(s) = -\frac{d}{ds}\mathcal{L}\left\{x(t)\right\} = -\frac{d}{ds}\int_0^\infty x(t)e^{-st}dt = -\int_0^\infty x(t)\frac{d}{ds}e^{-st}dt = -\int_0^\infty x(t)(-t)e^{-st}dt = \mathcal{L}\left\{tx(t)\right\}$$

Property 3

$$\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$$

Pf.

$$\mathcal{L}\left\{t^0\right\}=\mathcal{L}\left\{1\right\}=\int_0^\infty e^{-st}dt = \frac{e^{-st}|_0^\infty}{-s} = \frac{1}{s} = \frac{0!}{s^1}$$

$$\mathcal{L}\left\{t^1\right\} = \mathcal{L}\left\{t\cdot 1\right\} = -\frac{d}{ds}\mathcal{L}\left\{1\right\} = -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2} = \frac{1!}{s^2}$$

$$\mathcal{L}\left\{t^2\right\} = \mathcal{L}\left\{t\cdot t\right\} = -\frac{d}{ds}\mathcal{L}\left\{t\right\} = -\frac{d}{ds}\frac{1}{s^2} = \frac{2}{s^3} = \frac{2!}{s^3}$$

Assume $\mathcal{L}\left\{t^k\right\} = \frac{k!}{s^{k+1}}$, then apply property 2 again ,

$$\mathcal{L}\left\{t^{k+1}\right\} = \mathcal{L}\left\{t\cdot t^k\right\} = -\frac{d}{ds}\mathcal{L}\left\{t^k\right\} = -\frac{d}{ds}\frac{k!}{s^{k+1}} = \frac{(k+1)!}{s^{k+2}}$$

So the prove by Mathematical Induction is now complete.

2 Solving Zeroth Oder Bessel Differential Equation

2.1 General Bessel Differential Eqaution

$$t^2\frac{d^2y(t)}{dt^2}+t\frac{dy(t)}{dt}+\left(t^2-p^2\right)y=0 \qquad p\geq 0$$

p is called the order of Bessel's Equation

2.2 The solution of $p=0$, Zeroth Order

$$t^2\frac{d^2y(t)}{dt^2}+t\frac{dy(t)}{dt}+t^2y(t)=0$$

Divide the equation by t

$$ty" + y' + ty = 0$$

Apply Laplace Transform

$$\mathcal{L}\left\{ty"\right\}+\mathcal{L}\left\{y'\right\}+\mathcal{L}\left\{ty\right\}=0$$

Apply $\mathcal{L}$-property :

$$ty" \longleftrightarrow (-1)^1\frac{dF}{ds}\ ,\ \frac{d^2}{dt^2}y(t)\longleftrightarrow s^2Y(s)-sy(0)-y'(0) \text{ and } \frac{d}{dt}y(t)\longleftrightarrow sY(s)-y(0)$$

$$\Rightarrow \qquad -\frac{d}{ds}\mathcal{L}\left\{y"\right\}+\mathcal{L}\left\{y'\right\}-\frac{d}{ds}\mathcal{L}\left\{y\right\}=0$$

$$\Rightarrow \qquad -\frac{d}{ds}\left\{s^2Y(s)-sy(0)-y'(0)\right\}+\left\{sY(s)-y(0)\right\}-\frac{d}{ds}Y(s)=0$$

$$\Rightarrow \qquad \underbrace{-\frac{d}{ds}s^2Y(s)}_{2sY+s^2Y'}+\underbrace{\frac{d}{ds}sy(0)}_{+y(0)}+\underbrace{\frac{d}{ds}y'(0)}_{0}+sY(s)-y(0)-\frac{d}{ds}Y(s)=0$$

$$\Rightarrow \quad -2sY-s^2Y'+sY-Y'=0 \quad \Rightarrow \quad \left(s^2+1\right)\frac{dY(s)}{ds}+sY(s)=0 \quad \Rightarrow \quad \frac{dY(s)}{ds}+\frac{s}{s^2+1}Y(s)=0$$

$$\Rightarrow \quad \frac{dY(s)}{Y(s)}+\frac{s}{s^2+1}ds=0 \Rightarrow \quad \ln Y(s)+\frac{1}{2}\ln\left(s^2+1\right)=C' \quad \Rightarrow \quad \ln Y+\ln\left(s^2+1\right)^{\frac{1}{2}}=C$$

$$\Rightarrow \quad \ln\left[Y\sqrt{s^2+1}\right]=C' \quad \Rightarrow \quad Y(s)=\frac{C}{\sqrt{s^2+1}}$$

Let $C=1$, and rewrite it into the form

$$Y(s)=\left(s^2+1\right)^{-\frac{1}{2}}=\frac{1}{s}\left(1+\frac{1}{s^2}\right)^{-\frac{1}{2}}$$

Apply Binomial Expansion

$$Y(s) = \frac{1}{s}\sum_{k=0}^{\infty} C_k^{-\frac{1}{2}} \left(\frac{1}{s^2}\right)^k = \sum_{k=0}^{\infty} C_k^{-\frac{1}{2}} \frac{1}{s^{2k+1}}$$

Where

$$C_k^{\frac{-1}{2}} = \frac{\left(-\frac{1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)...\left(\frac{-1}{2}-k+1\right)}{k!} = \frac{(-1)^k\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)...\left(\frac{2k-1}{2}\right)}{k!}$$

$$= \frac{(-1)^k \overbrace{1.3.5...(2k-1)\cdot 2.4.6...2k}^{(2k)!}}{2^k k!\cdot \underbrace{2.4.6...2k}_{2^k k!}} = \frac{(-1)^k(2k)!}{2^k k! 2^k k!} = \frac{(-1)^k(2k)!}{2^{2k}(k!)^2}$$

$\therefore$

$$Y(s) = \sum_{k=0}^{\infty} C_k^{-\frac{1}{2}}\frac{1}{s^{2k+1}} = \sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{2^{2k}(k!)^2}\frac{1}{s^{2k+1}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}(k!)^2}\left(\frac{(2k)!}{s^{2k+1}}\right)$$

Since,

$$Y(s) = \mathcal{L}\{y(t)\} \text{ , which is the solution of } t^2\frac{d^2y(t)}{dt^2} + t\frac{dy(t)}{dt} + t^2 y = 0$$

Denote $y(t) = J_0(t)$, take the inverse Laplace Transfrom using the property

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$\therefore$

$$J_0(t) = \mathcal{L}^{-1}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}(k!)^2}\frac{(2k)!}{s^{2k+1}}\right\} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}(k!)^2}\mathcal{L}\left\{\frac{(2k)!}{s^{2k+1}}\right\} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}(k!)^2}t^{2k}$$

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