Example of Solving ODEs using Fourier and Laplace

February 20, 2017

The Problem

$$\frac{d^2y(x)}{dx^2} + 2y(x) = 3x$$

Ordinary Method using Wronskian : Skiped

Method using Laplace Transform

- Requirement : Initial conditions are given : $y(0) = y'(0) = 0$

$$\frac{d^2y(x)}{dx^2} + 2y(x) = 3x \qquad \mathcal{L} \qquad s^2Y(s) - sy(0) - y'(0) + 2Y(s) = \frac{3}{s^2}$$

With zero conditions $y(0) = y'(0) = 0$:

$$\left(s^2 + 2\right) Y(s) = \frac{3}{s^2}$$

Rearrange, and break it into partial fraction using Heaviside Cover Up method

$$Y(s) = \frac{3}{\left(s^2+2\right)s^2} = \frac{\left(\frac{3}{s^2+2}\right)_{s=0}}{s^2} + \frac{\left(\frac{3}{s^2}\right)_{s^2=2}}{s^2+2} = \frac{3}{2s^2} - \frac{3}{2\left(s^2+2\right)}$$

Rearrange it into standard form of Laplace Pair

$$Y(s) = \frac{3}{2}\frac{1}{s^2} - \frac{3}{2\sqrt{2}}\frac{\sqrt{2}}{\left(s^2+\sqrt{2}^2\right)}$$

So the solution, i.e. the inverse Laplace transform is

$$y(t) = \frac{3}{2}x - \frac{3}{2\sqrt{2}}\sin\sqrt{2}x$$

Method using Fourier Series

- Requirement : Range of $3x$ is given, e.g. $-\pi < x < \pi$

The Fourier Series of $3x$, $-\pi < x < \pi$ (Sawtooth Function)

$$f(x) = 3x \text{ is even : } \quad f(-x) = -3x = -f(x) \quad \Rightarrow a_n = 0 \; \forall n$$

$$3x = \sum_{n=0}^{\infty} b_n \sin nx$$

$$b_{n=}\frac{1}{\pi}\int_{-\pi}^{\pi} 3x \sin nx dx = \frac{-3}{\pi n}\int_{-\pi}^{\pi} x d\cos nx = \frac{-3}{\pi n}\underbrace{[x\cos nx]_{-\pi}^{\pi}}_{2\pi\cos nx} + \frac{3}{\pi n}\underbrace{\int_{-\pi}^{\pi}\cos nx dx}_{0} = \frac{6}{n}(-1)^{n+1}$$

$$3x = 6\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}$$

Now consider the original equation

$$\frac{d^2y(x)}{dx^2} + 2y(x) = 3x$$

It should have a solution in the form

$$y = \sum_{n=0}^{\infty}\beta_n \sin nx$$

Plug it into the equationd

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty}\beta_n \sin nx\right) + 2\left(\sum_{n=0}^{\infty}\beta_n \sin nx\right) = \left(6\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}\right)$$

$$\underbrace{\sum_{n=0}^{\infty}\left(-n^2\right)\beta_n \sin nx + 2\sum_{n=0}^{\infty}\beta_n \sin nx}_{} = \left(6\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}\right)$$

$$\sum_{n=0}^{\infty}\left(2-n^2\right)\beta_n \sin nx = \sum_{n=0}^{\infty}\left(\frac{6(-1)^{n+1}}{n}\right)\sin nx$$

Use Equality of Coefficient of Fourier Series

$$\left(2-n^2\right)\beta_n = \left(\frac{6(-1)^{n+1}}{n}\right) \qquad \beta_n = \frac{6(-1)^{n+1}}{\left(2-n^2\right)n}$$

So the solution is

$$y = 6\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\left(2-n^2\right)n}\sin nx$$

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