Solving PDE of 1^{st}order with constant coefficients

Andersen Ang

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$$a\frac{\partial u(x,y)}{\partial x} + b\frac{\partial u(x,y)}{\partial y} + cu(x,y) = f(x,y) \qquad a,b,c \in \mathbb{R}$$

First subtract the $cu(x,y)$ from both side, then "factorize" ∂u out

$$\partial u \left(\frac{a}{\partial x} + \frac{b}{\partial y} \right) = f(x,y) - cu(x,y)$$

Rearrange

$$\frac{f(x,y) - cu(x,y)}{\partial u} = \underbrace{\frac{a}{\partial x}}_{x-only} + \underbrace{\frac{b}{\partial y}}_{y-only}$$

The right hand size can be split into a equality

$$\frac{f(x,y) - cu(x,y)}{\partial u} = \frac{a}{\partial x} = \frac{b}{\partial y}$$

i.e.

$$\frac{\partial u}{f(x,y) - cu(x,y)} = \frac{\partial x}{a} = \frac{\partial y}{b}$$

Consider $\dfrac{\partial x}{a} = \dfrac{\partial y}{b}$

$$\frac{\partial x}{a} = \frac{\partial y}{b} \qquad \Longrightarrow \qquad bx - ay = c \quad \Longleftrightarrow \quad y = \frac{b}{a}x - \frac{c}{a}$$

Therefore

$$\frac{\partial u}{f(x,y) - cu(x,y)} = \frac{\partial x}{a}$$

Can be reduced to

$$\frac{\partial u}{f\left(x, \frac{b}{a}x - \frac{c}{a}\right) - cu\left(x, \frac{b}{a}x - \frac{c}{a}\right)} = \frac{\partial x}{a}$$

f and u are function of x only now, thus

$$\int a du = \int f(x) - cu(x) dx$$

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