Heat Equation

February 20, 2017

1D Diffusion

$$\frac{\partial u(x,t)}{\partial t} = \kappa \frac{\partial^2 u(x,t)}{\partial x^2}$$

Using separation of variable, let $u(x,t) = X(x)T(t)$, then

$$\frac{\partial X(x)T(t)}{\partial t} = \kappa \frac{\partial^2 X(x)T(t)}{\partial x^2}$$

Divide whole equation by $X(x)T(t)$

$$\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = \kappa \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}$$

Let the constant be λ, where $\lambda = -n^2$

$$\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = \kappa \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \lambda \quad \Longrightarrow \begin{cases} \dfrac{1}{T(t)}\dfrac{\partial T(t)}{\partial t} = \lambda \\ \\ \kappa \dfrac{1}{X(x)}\dfrac{\partial^2 X(x)}{\partial x^2} = \lambda \end{cases}$$

Re-arrange

$$\begin{cases} \dfrac{\partial T(t)}{\partial t} - \lambda T(t) = 0 \\ \\ \dfrac{\partial^2 X(x)}{\partial x^2} - \dfrac{\lambda}{\kappa} X(x) = 0 \end{cases} \Longleftrightarrow \begin{cases} \dfrac{\partial T(t)}{\partial t} + n^2 T(t) = 0 \\ \\ \dfrac{\partial^2 X(x)}{\partial x^2} + \dfrac{n^2}{\kappa} X(x) = 0 \end{cases}$$

The ODE with t is variable separable

$$\frac{\partial T(t)}{\partial t} + n^2 T(t) = 0 \qquad \Longrightarrow \frac{\partial T(t)}{T(t)} + n^2 \partial t = 0$$

$$\ln T(t) + n^2 t + C = 0$$

$$T(t) = e^{-n^2 t - C} = e^{-c} e^{-n^2 t} = A e^{-n^2 t}$$