The integration constant during separation of variation in solving PDE

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During solving the PDE using separation of variable, the following equation will appear (using shorthand notation)

$$\nabla^2\Phi(x,y)+k^2\Phi(x,y)=0$$

$$\frac{X"}{X}+\frac{Y"}{Y}+k^2=0$$

How to determine the integration constant ? There are 2 cases

$$\text{CaseI}\begin{cases}\frac{X"}{X(x)}=-k_x^2\\ \frac{Y"}{Y(y)}=-k_y^2\\ k_x^2+k_y^2=k^2\end{cases}\qquad\text{or}\qquad\text{Case II}\begin{cases}\frac{X"}{X(x)}=k_x^2\\ \frac{Y"}{Y(y)}=k_y^2\\ k_x^2+k_y^2=-k^2\end{cases}$$

Both case should give same solution

Integration constant case I

$$\begin{cases}X"+k_x^2X=0\\ Y"+k_y^2Y=0\\ k_x^2+k_y^2=-k^2\end{cases}$$

The general solution of $X"+k_x^2X=0$ is $X=A\sin k_xx+B\cos k_xx$, so

$$X=A_x\sin k_xx+B_x\cos k_xx\qquad Y=A_y\sin k_yy+B_y\cos k_yy$$

$$\Phi=XY=(A_x\sin k_xx+B_x\cos k_xx)(A_y\sin k_yy+B_y\cos k_yy)$$

A_x,A_y,B_x,B_y can be found using boundary conditions.

Integration constant case II

$$\begin{cases}X"-k_x^2X=0\\ Y"-k_y^2Y=0\\ k_x^2+k_y^2=k^2\end{cases}$$

The general solution of $X"-a^2X=0$ is $X=A\sinh k_xx+B\cosh k_xx$, so

$$X=A_x\sinh k_xx+B_x\cosh k_xx\qquad Y=A_y\sinh k_yy+B_y\cosh k_yy$$

$$\Phi=XY=(A_x\sinh k_xx+B_x\cosh k_xx)(A_y\sinh k_yy+B_y\cosh k_yy)$$

A_x,A_y,B_x,B_y can be found using boundary conditions.

Compare the two solution

$$\Phi = \begin{cases} (A_x \sinh k_x x + B_x \cosh k_x x)\,(A_y \sinh k_y y + B_y \cosh k_y y) \\ \\ (A_x \sin k_x x + B_x \cos k_x x)\,(A_y \sin k_y y + B_y \cos k_y y) \end{cases}$$

These 2 solution is actually the same.

$$\text{CaseI} \begin{cases} X" + k_x^2 X = 0 \\ Y" + k_y^2 Y = 0 \\ k_x^2 + k_y^2 = -k^2 \end{cases} \qquad \text{CaseII} \begin{cases} X" - k_x^2 X = 0 \\ Y" - k_y^2 Y = 0 \\ k_x^2 + k_y^2 = k^2 \end{cases}$$

Case II is same as case I , but how ?

If $X" + k_x^2 X = 0$ is changed into $X" - k_x^2 X$, that is **assuming that** $k_x(II)$ **is complex number while** $k_x(I)$ **is real number** , thus

$$-k_x^2(II) = +k_x^2$$

$$\Longleftrightarrow \pm j k_x(II) = \pm k_x(I)$$

Then by the fact that

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} \qquad \cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \qquad \cosh x = \frac{e^{x} + e^{-x}}{2}$$

Solution for case II can thus be transformed into case I.

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