Solving wave eqaution by separation of Variables in PDE

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1 Introduction

To solve the wave equation

$$\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi \qquad \Phi = \Phi(x,y,z,t) \ -\infty < t < +\infty$$

Apply *Fourier Transform* , $\nabla^2\Phi = \frac{1}{c^2}(j\omega)^2\Phi = -\frac{\omega^2}{c^2}\Phi = -\left(\frac{\omega}{c}\right)^2\Phi$
This become *Helmholtz Equation*

$$\nabla^2\Phi + k^2\Phi = 0$$

- $k = \frac{\omega}{c}$: wave number
- $\Phi = \Phi(x,y,z,\omega) = \int_{-\infty}^{\infty}\phi(x,y,z,t)e^{-j\omega t}dt$

2 Separation of Variable in Cartesian Coordinate

$$\nabla^2\Phi(x,y,z,t) = \frac{1}{c^2}\frac{\partial^2\Phi\left(x,y,z,t\right)}{\partial t^2}$$

Separation of Variable , let

$$\Phi(x,y,z,t) = X(x)Y(y)Z(z)T(t)$$

Then it becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)X(x)Y(y)Z(z)T(t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}X(x)Y(y)Z(z)T(t)$$

That is

$$Y(y)Z(z)T(t)\frac{\partial^2X(x)}{\partial x^2} + X(x)Z(z)T(t)\frac{\partial^2Y(y)}{\partial y^2} + X(x)Y(y)T(t)\frac{\partial^2Z(z)}{\partial z^2} = \frac{X(x)Y(y)Z(z)}{c^2}\frac{\partial^2T(t)}{\partial t^2}$$

It become ordinary derivatives : $\frac{\partial f(x)}{\partial x} = \frac{df(x)}{dx}$

$$Y(y)Z(z)T(t)\frac{d^2X(x)}{dx^2} + X(x)Z(z)T(t)\frac{d^2Y(y)}{dy^2} + X(x)Y(y)T(t)\frac{d^2Z(z)}{dz^2} = \frac{X(x)Y(y)Z(z)}{c^2}\frac{d^2T(t)}{dt^2}$$

Using short hand notation

$$YZTX" + XZTY" + XYTZ" = \frac{XYZ}{c^2}T"$$

Divide whole equation by $XYZT$

$$\frac{X"}{X} + \frac{Y"}{Y} + \frac{Z"}{Z} = \frac{1}{c^2}\frac{T"}{T}$$

$$\Rightarrow \begin{cases} \dfrac{X"}{X} = \underbrace{\dfrac{1}{c^2}\dfrac{T"}{T} - \dfrac{Y"}{Y} - \dfrac{Z"}{Z}}_{x-\text{independent}} = -p^2 \\ \dfrac{Y"}{Y} = \underbrace{\dfrac{1}{c^2}\dfrac{T"}{T} - \dfrac{X"}{X} - \dfrac{Z"}{Z}}_{y-\text{independent}} = -q^2 \\ \dfrac{Z"}{Z} = \underbrace{\dfrac{1}{c^2}\dfrac{T"}{T} - \dfrac{Y"}{Y} - \dfrac{Z"}{Z}}_{z-\text{independent}} = -r^2 \\ \dfrac{1}{c^2}\dfrac{T"}{T} = \underbrace{\dfrac{X"}{X} + \dfrac{Y"}{Y} + \dfrac{Z"}{Z}}_{t-\text{independent}} = -s^2 \end{cases} \Rightarrow \begin{cases} X" + p^2X = 0 \\ Y" + q^2Y = 0 \\ Z" + r^2Z = 0 \\ T" + c^2s^2T = 0 \end{cases} \quad p, q, r, s \text{ are separation constant}$$

Then the general solutions are

$$\begin{cases} X(x) = Ae^{jpx} + Be^{-jpx} \\ Y(y) = Ce^{jqy} + De^{-jqy} \\ Z(z) = Ee^{jrz} + Fe^{-jrz} \\ T(t) = Ge^{jcst} + He^{-jcst} \end{cases} \quad \begin{array}{c}\text{A,B,...,H are constants} \\ \text{that determined by using} \\ \text{Boundary Conditions}\end{array}$$

The two exponential terms are related to forward propagation and backward propagation, suppose that select the forward propagation

$$X(x) = Ae^{jpx} \qquad Z(z) = Ee^{jrz}$$

$$Y(y) = Ce^{jqy} \qquad T(t) = He^{-jcst}$$

By Principle of Superposition, the final solution of the equation then is

$$\Phi(x, y, z, t) = X(x)Y(y)Z(z)T(t) = ACEH\, e^{jpx+jqy+jrz-jcst}$$

$$\Phi(x, y, z, t) = \rho e^{j(px+qy+rz-cst)}$$

Where $\rho = ACEH$

The p, q, r are actually the x, y, z component of the wave vector $\mathbf{k}$, that $s = |\mathbf{k}| = k = \frac{2\pi}{\lambda}$, then $cs = f\lambda\frac{2\pi}{\lambda} = \omega$

$$\Phi(x, y, z, t) = \rho \exp\left[j\left(k_x x + k_y y + k_z z - \omega t\right)\right]$$

$$\Phi(x, y, z, t) = \rho e^{j(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

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