A special 2^{nd} order Constant Coefficient ODE

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When solving some PDEs using separation of variables, it will yield the following ODE :

$$\frac{d^2y(x)}{dx^2} = Cy(x)$$

The basic idea on how to solve it is to assume solution in the fom $Ae^{\lambda x}$ exists and try to solve for the unknowns λ

Case I. $C = k^2$

$$\frac{d^2y(x)}{dx^2} = k^2y(x)$$

or

$$\frac{d^2y(x)}{dx^2} - k^2y(x) = 0$$

Let the solution be $y(x) = Ae^{\lambda x}$, put this solution into the equation

$$\frac{d^2}{dx^2}\left(Ae^{\lambda x}\right) - k^2\left(Ae^{\lambda x}\right) = 0$$

i.e.

$$Ae^{\lambda x}(\lambda^2 - k^2) = 0$$

Since $Ae^{\lambda x} = y(x)$ can not be zero (otherwise it is meaningless), so

$$\lambda^2 = k^2$$

Or

$$\lambda = \pm k$$

Case II. $C = -k^2$

$$\frac{d^2y(x)}{dx^2} = -k^2y(x)$$

or

$$\frac{d^2y(x)}{dx^2} + k^2y(x) = 0$$

Let the solution be $y(x) = Ae^{\lambda x}$, put this solution into the equation

$$\frac{d^2}{dx^2}\left(Ae^{\lambda x}\right) + k^2\left(Ae^{\lambda x}\right) = 0$$

i.e.

$$Ae^{\lambda x}(\lambda^2 + k^2) = 0$$

Since $Ae^{\lambda x} = y(x)$ can not be zero (otherwise it is meaningless), so

$$\lambda^2 = -k^2$$

Or

$$\lambda = \pm jk$$

Case III. $C = 0$

$$\frac{d^2y(x)}{dx^2} = 0$$

Then it is easy to solve via direct integration

$$\frac{dy(x)}{dx} = C_1$$

$$y(x) = C_1 x + C_2$$

The solution listed above only contain one term, but actually the complete solution consists of two terms, thus let's consider complete solution

Case I. $C = k^2$

$$\frac{d^2y(x)}{dx^2} = k^2 y(x)$$

or

$$\frac{d^2y(x)}{dx^2} - k^2 y(x) = 0$$

Let the solution be $y(x) = Ae^{\lambda x} + Be^{-\lambda x}$, put this solution into the equation

$$\frac{d^2}{dx^2}\left(Ae^{\lambda x} + Be^{-\lambda x}\right) - k^2\left(Ae^{\lambda x} + Be^{-\lambda x}\right) = 0$$

i.e.

$$A\lambda^2 e^{\lambda x} + B\lambda^2 e^{-\lambda x} - k^2\left(Ae^{\lambda x} + Be^{-\lambda x}\right) = 0$$

Extract out $Ae^{\lambda x} + Be^{-\lambda x}$

$$\left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(\lambda^2 - k^2\right) = 0$$

Again, since $Ae^{\lambda x} + Be^{-\lambda x} = y(x)$ can not be zero (otherwise it is meaningless), so

$$\lambda^2 = k^2$$

Or

$$\lambda = \pm k$$

And therefore the solution is

$$y(x) = Ae^{kx} + Be^{-kx}$$

Or we can express the solution in terms of hyperbolic *sinh* and *cosh*

A small fact :

For two number A, B , there exists another pair of number P , Q such that

$$P + Q = A \text{ and } P - Q = B$$

A "simple proof "

$$A = \frac{A + A + B - B}{2} = \frac{A + B + A - B}{2} = \frac{A + B}{2} + \frac{A - B}{2}$$

$$B = \frac{A - A + B + B}{2} = \frac{A + B + (-A) - (-B)}{2} = \frac{A + B}{2} - \frac{A - B}{2}$$

Where

$$P = \frac{A + B}{2} \qquad Q = \frac{A - B}{2}$$

Therefore

$$y(x) = Ae^{kx} + Be^{-kx}$$

is same as

$$y(x) = (P + Q)e^{kx} + (P - Q)e^{-kx}$$

$$= P\left(e^{kx} + e^{-kx}\right) + Q\left(e^{kx} - e^{-kx}\right)$$

$$= \underbrace{2P}_{N}\left(\frac{e^{kx} + e^{-kx}}{2}\right) + \underbrace{2Q}_{M}\left(\frac{e^{kx} - e^{-kx}}{2}\right)$$

$$= N\cosh kx + M\sinh kx$$

Therefore

$$\frac{d^2y(x)}{dx^2} - k^2y(x) = 0 \qquad \Longrightarrow \qquad y(x) = \begin{cases} Ae^{kx} + Be^{-kx} \\ or \\ N\cosh kx + M\sinh kx \end{cases}$$

Case II. $C = -k^2$

$$\frac{d^2y(x)}{dx^2} = -k^2 y(x)$$

or

$$\frac{d^2y(x)}{dx^2} + k^2 y(x) = 0$$

Let the solution be $y(x) = Ae^{\lambda x} + Be^{-\lambda x}$, put this solution into the equation

$$\frac{d^2}{dx^2}\left(Ae^{\lambda x} + Be^{-\lambda x}\right) + k^2\left(Ae^{\lambda x} + Be^{-\lambda x}\right) = 0$$

i.e.

$$A\lambda^2 e^{\lambda x} + B\lambda^2 e^{-\lambda x} + k^2\left(Ae^{\lambda x} + Be^{-\lambda x}\right) = 0$$

Extract out $Ae^{\lambda x} + Be^{-\lambda x}$

$$\left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(\lambda^2 + k^2\right) = 0$$

Again, since $Ae^{\lambda x} + Be^{-\lambda x} = y(x)$ can not be zero (otherwise it is meaningless), so

$$\lambda^2 = -k^2$$

Or

$$\lambda = \pm jk$$

And therefore the solution is

$$y(x) = Ae^{jkx} + Be^{-jkx}$$

Or we can express the solution in terms of sin and cos

$$y(x) = (P+Q)e^{jkx} + (P-Q)e^{-jkx}$$

$$= P\left(e^{jkx} + e^{-jkx}\right) + Q\left(e^{jkx} - e^{-jkx}\right)$$

$$= 2P\left(\frac{e^{jkx} + e^{-jkx}}{2}\right) + 2jQ\left(\frac{e^{jkx} - e^{-jkx}}{2j}\right)$$

$$= M\cos kx + jN\sin kx$$

Or using Euler's Equation

$$e^{jkx} = \cos kx + j\sin kx$$

$$y(x) = Ae^{jkx} + Be^{j(-kx)}$$

$$= A\left(\cos kx + j\sin kx\right) + B\left(\cos(-kx) + j\sin(-kx)\right)$$

$$= \underbrace{(A+B)}_{2P} \cos kx + j \underbrace{(A-B)}_{2Q} \sin kx$$

$$= M \cos kx + jN \sin kx$$

Therefore

$$\frac{d^2y(x)}{dx^2} + k^2 y(x) = 0 \qquad \Longrightarrow \qquad y(x) = \begin{cases} Ae^{jkx} + Be^{-jkx} \\ or \\ M \cos kx + jN \sin kx \end{cases}$$

Case III. $C = 0$

$$\frac{d^2y(x)}{dx^2} = 0$$

Then it is easy to solve via direct integration

$$\frac{dy(x)}{dx} = C_1$$

$$y(x) = C_1 x + C_2$$

$$-END-$$