Derivation of Wave Equation and Heat Equation

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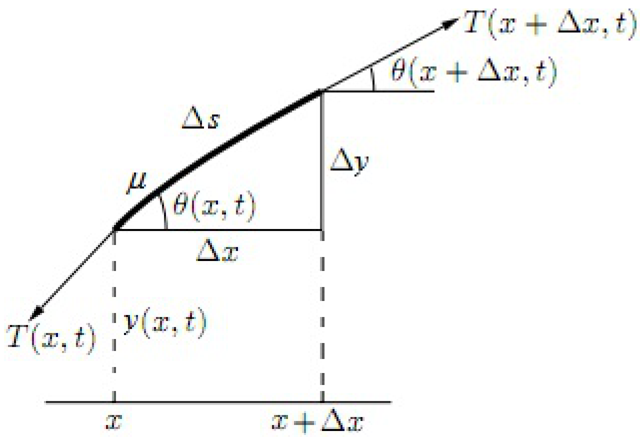
2011-10-7

Wave Equation



For one Dimensional Wave

$$Y = y(x,t)$$

The net upward force is

$$\begin{aligned} T(x+\Delta x,t) - T(x,t) &= T\sin\theta_{x+\Delta x} - T\sin\theta_x \\ &= T\left(\sin\theta_{x+\Delta x} - \sin\theta_x\right) \end{aligned}$$

For a small vibration,

$$\Delta x \to 0 \implies \theta \to 0 \iff \sin\theta \simeq \tan\theta \simeq \theta$$

Also ,

$$\tan\theta = \lim_{\Delta x \to 0} \frac{\Delta y(x,t)}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x,t) - y(x,t)}{\Delta x} = \frac{\partial y}{\partial x}$$

Therefore ,

$$\sin\theta \simeq \frac{\partial y}{\partial x}$$

Hence , the net upward force is

$$T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right)$$

By Newton's Second Law, net external force = mass×acceleration = length× linear mass density× acceleration

$$T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) = F = \Delta s \cdot \mu \cdot \left(\frac{\partial^2 y}{\partial t^2} + \epsilon\right)$$

As the vibration is small , $\Delta s \simeq \Delta x$

$$T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) = \mu\Delta x \left(\frac{\partial^2 y}{\partial t^2} + \epsilon\right)$$

Re-arrange the terms

$$\frac{T}{\mu} \frac{\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}}{\Delta x} = \frac{\partial^2 y}{\partial t^2} + \epsilon$$

Take the limit

$$LHS \; : \; \lim_{\Delta x \to 0} \frac{\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}}{\Delta x} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) \qquad RHS \; : \; \lim_{\Delta x \to 0} \epsilon = 0$$

Thus

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y(x,t)}{\partial t^2} \qquad c = \sqrt{\frac{T}{\mu}}$$

For the 3D case

$$U = u(x, y, z, t)$$

The Generalized Wave Equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Boundary Condition for 1D String with length L

- If both end of the string are fixed : $y(0, t) = y(L, t) = 0 \; \forall t > 0$
- Initial shape of the string describe by function $f(x)$: $y(x, 0) = f(x) \;\; x \in [0, L]$
- Initial velocity of the string describe by function $g(x)$: $\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = g(x) \;\; x \in [0, L]$
- Boundedness : $\left|y(x,t)\right| < M \;\; x \in [0, L] \; , \; t > 0$

For Vairable Tension in the string

$$T \to T(x)$$

The net upward force is then

$$T(x)\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) = \mu \Delta x \left(\frac{\partial^2 y}{\partial t^2} + \epsilon\right)$$

$$\frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right) = \mu \frac{\partial^2 y}{\partial t^2}$$

Heat Eqaution

The temperature at a point (x, y, z) of a solid at time t is $u(x, y, z, t)$ and let κ be the thermal conductivity, σ be the specific heat and τ be density of solid, assumed constant, then

$$\frac{\partial u}{\partial t} = \eta \nabla^2 u \qquad \text{where } \eta = \frac{\kappa}{\sigma\tau}$$

Let V be arbitary volume within the solid, S be surface. Total heat flux across S (amount of heat leaving S per unit time) :

$$\iint_S (-\kappa \nabla u) \cdot d\mathbf{S}$$

Thus, using Gauss's Divergence Theorem, amount of heat entering S per unit time is

$$\iint_S (\kappa \nabla u) \cdot d\mathbf{S} = \iiint_V \nabla \cdot (\kappa \nabla u)\, dV$$

The heat contained in a volume V is

$$Q = \iiint_V \sigma\tau u dV$$

And the time rate of increase of heat is

$$\frac{\partial Q}{\partial t} = \frac{\partial}{\partial t} \iiint_V \sigma\tau u dV = \iiint_V \sigma\tau \frac{\partial u}{\partial t} dV$$

time rate of increase of heat =amount of heat entering S per unit time

$$\iiint_V \sigma\tau \frac{\partial u}{\partial t} dV = \iiint_V \nabla \cdot (\kappa \nabla u)\, dV$$

So

$$\sigma\tau \frac{\partial u}{\partial t} = \nabla \cdot (\kappa \nabla u) \qquad \Rightarrow \frac{\partial u}{\partial t} = \frac{\kappa}{\sigma\tau} \nabla^2 u$$

$\therefore$

$$\frac{\partial u}{\partial t} = \eta \nabla^2 u \qquad \text{where } \eta = \frac{\kappa}{\sigma\tau}$$

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