Some points about algebra of vector and matrix

Andersen Ang

Firest created: before 2013. Last updated: 2017-Feb-1

1 Vector Algebra

There are 3 type of quantities : *Scalar*, only magnitude. (Temperature, Speed, Mass ...). *Vector*, magnitude + direction. (Velocity, Force, ...). *Tensor*, but what is a tensor? :)

1.1 Vector

$\mathbf{v} = \overrightarrow{OP} = \overline{OP} = x\mathrm{i} + y\mathrm{j} + z\mathrm{k} = x\widehat{x} + y\widehat{y} + z\widehat{z} = x\hat{e_1} + y\hat{e_2} + z\hat{e_3} = (x, y, z) = \langle x, y, z \rangle = |OP|\widehat{OP} = \Big| OP \Big\rangle$

- Need a reference corrdinate system.
- For point $A = (a_1, a_2, ..., a_n)$ and point $B = (b_1, b_2, ..., b_n)$, the vector pointing from A to B is $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, ..., b_n - a_n)$.
- The concept of free vector (translation of vectors).
- Magnitude. For vector $\overrightarrow{OP} = (p_1, p_2, ..., p_n)$, magnitude $\left\| \overrightarrow{OP} \right\| = \sqrt{p_1^2 + p_2^2 + ... + p_n^2}$.
- Direction. For two-dimensional vector $\mathbf{v} = (v_1, v_2)$, $\theta = \tan^{-1}\frac{v_2}{v_1}$.
- Negative. For $u = -v$, $u = (-v_1, -v_2, ..., -v_n)$. u is anti-parallel to v. They have same magnitude but opposite direction.
- Vector equality 1 (geometric point of view). $\mathbf{v} = \mathbf{u}$ iff they have same magnitude **and** same direction.
- Vector equality 2 (component point of view). $\mathbf{v} = \mathbf{u}$ iff all their components are equal : $v_1 = u_1$, $v_2 = u_2$, ...
- Scalar multiplication. For $u = kv$, $u = (ku_1, ku_2, ..., ku_n)$. u parallel to v, with magnitude is k times larger.
- Addition. For $w = u + v$, $w = (u_1 + v_1, u_2 + v_2, ...u_n + v_n)$. The addition obey parallelogram rule.

1.2 Vector Operation

1.2.1 Dot Product / Inner Product / Scalar Product

Definition. Dot product between two vectors $a$and b , is a scalar. For example, for three-dimensional vectors a, b

$$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3 = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3)$$

$$a \cdot b = |a||b|\text{cos}\theta_{ab} \quad 0 \leq \theta \leq \pi$$

Corollary. *Angle between two vector*

$$cos\theta = \frac{a \cdot b}{|a||b|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}} \quad a, b \neq 0$$

Corollary. *u, v perpendicular to each other* $\iff u \cdot v = 0$.

1.2.2 Cross Product / Outter Product / Vector Product

Definition. Cross Product is a vector that perpendicular to the plane formed by the two vector

$$a \times b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a \times b = |a||b|\text{sin}\theta_{ab}\hat{n}$$

2 Matrix Algebra

2.1 Determinant

For a square matrix$A_{n\times n}$, the **determinant of A,** *det A* , is a number :

$$\text{det}A = \underbrace{\sum_{i=1}^{n} a_{ij}(-1)^{i+j} det\, M_{ij}}_{\text{Column Expansion}} = \underbrace{\sum_{j=1}^{n} a_{ij}(-1)^{i+j} det\, M_{ij}}_{\text{Row Expansion}}$$

The $\pm$Sign $(-1)^{i+j}$

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The Minor M_{ij} is a $(n-1) \times (n-1)$ submatrix formed by deleting i^{th} row and j^{th} column form $A_{n\times n}$

So for a $n \times n$ determinant, after $n-1$ times of reduction, it reduce into a number.

Example. For 2×2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$,

$$\text{det}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}(-1)^{1+1}\text{det}M_{11} + a_{21}(-1)^{2+1}\text{det}M_{21} = a_{11}a_{22} - a_{21}a_{12}.$$

Example. 3×3 , $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$
$= a_{11}(-1)^{1+1}\text{det}M_{11} + a_{12}(-1)^{1+2}\text{det}M_{12} + a_{13}(-1)^{1+3}\text{det}M_{13}$
$= a_{11}\text{det}\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\text{det}\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\text{det}\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$
Apply the 2×2 Corollary ,
$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$
$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

Corollary. *(**Rule of Sarrus**) or matrix* $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aef + bfg + cdh - ahf - bdi - gec$$

2.2 Properties of Determinant

1. Transpose $\det M^T = \det M$
2. Interchange row / column, add a negative sign
3. For any triangular matrix B , $\text{det}B = \text{diag}B$
4. $\text{det}AB = \text{det}A\text{det}B$
5. (Generalization of 4.) $\det \left(\prod_{i=1}^{i=n} A_i\right) = \prod_{i=1}^{i=n} (\det A_i)$

2.3 Application of Determinant

- **Test the existance of inverse.** For square matrix $A_{n\times n}$, if $\text{det}A = 0$, A does not have inverse. i.e. A is a singular matrix

  i.e.

  $$\det A = 0 \iff A^{-1}\text{does not exist} \iff A\vec{x} = \vec{0} \text{ has non-zero solution}$$

- **Find eigenvalue.** Eigenvalue λ for matrix A can be found by the eqaution : $\det(A_{n\times n} - \lambda I_n) = 0$
- **Compute cross product.** $\vec{a} \times \vec{b} = \textbf{det} \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$**.**

3 Eigenvalue & Eigenvector

Definition. For matrix $A_{n\times n}$, v is a eigenvector of A with eigenvalue λ iff

$$Av = \lambda v \quad , \ v \neq 0$$

$(A_{n\times n} - \lambda I_n)\, v = 0$ has non-zero solution iff matrix $(A - \lambda I)$ do not have inverse. i.e. $\det(A - \lambda I) = 0$.

So eigenvalues of A are determined by

$$\det(A - \lambda I) = 0$$

Expansion of such determinant give a **characteristic polynomial** equation in degree n with exactly n roots (including complex root and repeated roots).

Then ,with λ, the eigenvector v can be determined by

$$(A - \lambda I)\, v = 0$$

Each eigenvalue has at least one corresponding eigenvector. If eigenvalue λ has multiplicity m , then it may have from 1 to m lineary independent eigenvector

- If A is in order 2 , $\det(A - \lambda I) = \lambda^2 - (\text{Tr}A)\,\lambda + \det A$
- If A is ino order 3, $\det(A - \lambda I) = -\lambda^3 + (\text{Tr}A)\,\lambda^2 - \left(\sum_{i=1}^{i=3} \det M_i\right)\lambda + \det A$. Where$M_i$ is the minor of A.
- Hence for triangular $A_{2\times 2}$: $\lambda^2 - (\text{Tr}A)\,\lambda = 0 \iff \lambda\,(\lambda - \text{Tr}A) = 0 \iff \lambda = \text{Tr}A$.
- Hence for triangular $A_{3\times 3}$: $-\lambda^3 + (\text{Tr}A)\,\lambda^2 - \left(\sum_{i=1}^{i=3} \det M_i\right)\lambda = 0$
- If A is triangular, then eigenvalues are the **diagonal entries** of the matrix

Definition. The matrix $A_{n\times n}$, the set of all eigenvectors corresponding to λ, together with zero vector, is a subspace of $\mathbb{R}^n$, the **eigenspace** of λ.