Positive Definite Matrix

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A matrix $\mathbf{A}$ is positive definite (denoted as $\mathbf{A} > 0$)if

$$\mathbf{x^T A x} > \mathbf{0} \qquad \forall \mathbf{x} \neq \mathbf{0}$$

1 Properties of positive definite matix

1.1 A is positive definite if $\lambda_i(\mathbf{A}) > 0$, $\forall i$

If a matrix is positive definite, then all the eigenvalues of that matrix is positive

$$\mathbf{x^T A x} = \mathbf{x^T} \lambda \mathbf{x} = \lambda \mathbf{x^T x}$$

$$\Longrightarrow \lambda = \frac{\mathbf{x^T A x}}{\mathbf{x^T x}} > 0$$

Since $\mathbf{x^T x} = \sum x_i^2 \geq 0$ and $\mathbf{x^T A x}$ >0

1.2 All diagonal element > 0 is a condition for $\mathbf{A} > \mathbf{0}$

Consider diagonal matrix

$$D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Recall the eigenvalue of diagonal matrix is the diagonal

$$\begin{aligned} & \det(D - \lambda I) = 0 \\ \Longleftrightarrow \quad & \det \begin{bmatrix} a - \lambda & \\ & b - \lambda \end{bmatrix} = 0 \\ \Longleftrightarrow \quad & (\lambda - a)(\lambda - b) = 0 \\ \Longleftrightarrow \quad & \lambda = a \,\text{or}\, b \end{aligned}$$

Since for positive definite matrix, all the eigenvalue should be positive, thus

If $\mathbf{A}$ is positive definite matrix , $a_{ii} > 0$
It means $a_{ii} > 0$ is codition for $\mathbf{A}$ is positive definite, the inverse.

1.3 $\mathbf{A} > \mathbf{0}$ iff $\mu_i(\mathbf{A}) > 0$

Recall that *minor* of order k is a submatrix by deleting $n-k$ row and column.

Principal minor (denote as $\mu(A)$) is minor that the last $n-k$ row and column are deleted.

e.g.

$$A = \begin{bmatrix} 1 & 3 & -2 & -1 \\ 1 & 2 & 4 & -6 \\ 2 & 3 & 3 & 4 \\ 0 & -2 & -1 & 4 \end{bmatrix}$$

All the principal minors are

$$\mu_1(A) = [a_{11}] = [1] \qquad \mu_2(A) = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \qquad \mu_3(A) = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 2 & 4 \\ 2 & 3 & 3 \end{bmatrix} \qquad \mu_4(A) = A$$

Theorem
If A is positive definite , then all the principal minor is also positive definite

2 Determine the definiteness of a matrix

Given a matrix A , how to know if it is positive definite ?

Method 1. Find all the eigenvalue of A , if $\lambda_i(A) > 0$ then A is positive definite.

Method 2. Find all determinant of principal of A , if all > 0 , then A is positive definite.

Method 3. (To show A is not positive definite). If there is a element in the diagonal < 0 , that matrix is not positive definite.

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