Quick Review on Linear Space

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1 Linear Space Definition

The set V is a linear space if

1. Addition is commutative : $\forall u, v \in V$, $u + v = v + u$
2. Addition is associative : $\forall u, v, w \in V$, $(u + v) + w = u + (v + w) = v + (u + w)$
3. Identity element in addition : $\exists!\ O \in V$ s.t. $\forall u \in V$, $u + O = u$
4. Negative : $\exists!\ v \in V$ s.t. $\forall u \in V$, $u + v = O$
5. Scalar multiplication is communtative, associative, distributive : $\forall a, b \in \mathbb{R}$ and $\forall u, v \in V$, $a(bu) = (ab)u = b(au)$, $(a + b)u = au + bu$, $a(u + v) = au + av$
6. Identity element in scalar multiplication : $\exists!\ 1 \in \mathbb{R}$ s.t. $\forall u \in V$, $1u = u$

2 Subspace

For nonempty subset S of linear space V , if S is

1. Close in addition : $\forall u, v \in S$, then $u + v \in S$
2. Close in scalar multiplication : $\forall a, u \in S$, then $au \in S$

Then S is a subspace of V

3 Linear Combination, Linear Independence & Dependence

For vector u, v , their linear combination w, for any value of a and b is $w = av + bu$

For a linear space V , a set of vector v_1 , v_2 , ... , v_n is linear dependence if there exists a group of non-zero element a_1, a_2 , ... , a_n in $\mathbb{R}$ that

$$\sum_{i=1}^{n} a_i v_i = O$$

If there is no such set of $\{a_i\}$, then $\{v_i\}$ are all linear independent

Geometrically, it means if a vector can not be expressed as a linear combination of other vectors, then such vector is linear independent to other vector, and if such vector can be expressed as a linear combination of other vectors, then such vector is linear dependent to other vector.

4 Span, Basis, Dimension

A vector w is span by u, v if w can be expressed as a linear combination of u and v

Geometrically, that means w can be represented / generated / formed by $au + bv$ for some a, b

For a set of vectors $\{v_i\}_{i=1}^{i=n}$ in a linear space V , if

1. v_1 , v_2 , ... , v_n are all linearly independent
2. v_1 , v_2 , ... , v_n span V : span$\{v_1, v_2, ..., v_n\} = V$

Then $\{v_i\}_{i=1}^{i=n}$ is a basis for V

For linear space V , if V has a basis of n vectors, then dimension of V is n

5 Kernel / Null space

The kernel , or the null space of a matrix A is a set that it contains all the element x s.t. $Ax = 0$

That is , $Ker(A) = \{\, x \in V \, : \, Ax = 0 \,\}$

6 Rank

For a matrix A ,

$$\begin{aligned} \text{Column rank}(A) &= \text{Row rank}(A) = \text{Dimension of largest non-singular submatrix of } A \\ &= \text{Number of pivot of } A \end{aligned}$$

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