The L^p Norm of Vector

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1. What is L^p-norm ?

Norm is a kind of measure of the size of an mathematical object.

Ordering exists for rational number (and real number): we can compare the size of the rational number esily.
For example, we know "7" is larger than "4" , and "0" is larger than "$-\frac{3}{2}$"

For complex number, there is no ordering.
It is wrong to say $3i > -2i$, $-3i < -2i$.

Then how to compare the size of two complex number ?
From knowledge of complex number, the *modulus* is used.
For $x + iy$ and $a + ib$, we compare $\sqrt{x^2 + y^2}$ and $\sqrt{a^2 + b^2}$

In the same way, the size of a 3D vector is $|v| = \sqrt{v_x^2 + v_y^2 + v_z^2}$
Thus it leads to the following definition :

2. The L^p-Norm of vector

DEFINITION For a vector $x = [x_1 , x_2 , \dots , x_n]^T$, and for $0 < p < \infty$, the $p-$norm are defined as

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

Example : The 1-norm

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

The 2-norm

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

The $\infty-$norm

For the ∞-norm, a limit definition is used

$$\|x\|_\infty = \lim_{p \mapsto \infty} \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

Theorem

It is interesting that, the ∞-norm is actually equal to the max element !

$$\|x\|_\infty = \max_{1 \leq j \leq n} \{x_1, x_2, ..., x_n\}$$

Proof.

Part 1. Extract the max element out

First consider the definition of $\infty-$norm

$$\|x\|_\infty = \lim_{p \mapsto \infty} \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

WLOG, let x_j be the largest element in the vector, expand the summation notation

$$\|x\|_\infty = \lim_{p \mapsto \infty} (|x_1|^p + |x_2|^p + ... + |x_j|^p + ... + |x_n|^p)^{\frac{1}{p}}$$

Factorize x_j out

$$\|x\|_\infty = \lim_{p \mapsto \infty} \left(x_j^p \left[\left| \frac{x_1}{x_j} \right|^p + ... + 1 + ... + \left| \frac{x_n}{x_j} \right|^p \right] \right)^{\frac{1}{p}}$$

$$\|x\|_\infty = x_j \left(\lim_{p \mapsto \infty} \left[\left| \frac{x_1}{x_j} \right|^p + ... + 1 + ... + \left| \frac{x_n}{x_j} \right|^p \right]^{\frac{1}{p}} \right)$$

Then, we have factored x_j out.

The remaining work is to deal with the ( ) term.

Part 2. Showing that the limit is in the form 0^0 , indetermined form

Since x_j is the largest element in the vector, thus

$$\frac{x_i}{x_j} < 1$$

And thus

$$\left| \frac{x_i}{x_j} \right|^p \longrightarrow 0 \qquad \text{for } p \to \infty$$

Also

$$\left[\left| \frac{x_1}{x_j} \right|^p + ... + 1 + ... + \left| \frac{x_n}{x_j} \right|^p \right]^{\frac{1}{p}} \longrightarrow 0^0 \qquad \text{for } p \longrightarrow \infty$$

Which is an indetermined form.

Part 3. Apply L' Hospital rule to evaulate the limit
(The following is a old trick in HK A-Level pure mathematics to tackle 0^0)
To handle this indetermined form, consider

$$z = \frac{1}{y}^{\frac{1}{y}}$$

Thus

$$\lim_{y \mapsto \infty} z = 0^0$$

In stead of evaluate the limt directly, consider the limit of $\ln z$, first, take ln

$$\text{lnz}=\frac{1}{y} \ln \frac{1}{y} = \frac{-1}{y} \ln y$$

Thus

$$\lim_{y \mapsto \infty} \ln z = - \lim_{y \mapsto \infty} \frac{\ln y}{y} \qquad \left(\text{in the form } \frac{\infty}{\infty}, \text{ still indetermined form} \right)$$

But now the indetermined form is in the form $\dfrac{f(y)}{g(y)}$, we can now apply L'Hospital Rule

$$- \lim_{y \mapsto \infty} \frac{\ln y}{y} = - \lim_{y \mapsto \infty} \frac{1/y}{1} = 0$$

Thus

$$\lim_{y \mapsto \infty} \ln z = 0$$

And therefore

$$\lim_{y \mapsto \infty} z = e^{\lim_{y \mapsto \infty} \ln z} = 1$$

Thus

$$\lim_{y \mapsto \infty} \frac{1}{y}^{\frac{1}{y}} = 1$$

Part 4. Finishing the proof
And thus

$$\lim_{p \mapsto \infty} \left[\left| \frac{x_1}{x_j} \right|^p + ... + 1 + ... + \left| \frac{x_n}{x_j} \right|^p \right]^{\frac{1}{p}} = 1$$

And therefore

$$\|x\|_\infty = \lim_{p \mapsto \infty} \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} = x_j \left(\lim_{p \mapsto \infty} \left[\left| \frac{x_1}{x_j} \right|^p + ... + 1 + ... + \left| \frac{x_n}{x_j} \right|^p \right]^{\frac{1}{p}} \right)$$

$$\|x\|_\infty = x_j \cdot 1 = \max_{1 \leq j \leq n} \{x_1, x_2, ..., x_n\}$$

□

Remark 1. The extraction of the max element x_j is to apply $\dfrac{x_i}{x_j} < 1$.

Remark 2. If there are more than one max element in the vector, for example, $x = [1, 2, 5, 11, 11, 6, 9]^T$, such theorem still hold. Because the limit

$$\lim_{p \mapsto \infty} \left[\left| \frac{x_1}{x_j} \right|^p + ... + 1 + 1 + ... + \left| \frac{x_n}{x_j} \right|^p \right]^{\frac{1}{p}} = \lim_{p \mapsto \infty} \left[\sum \delta_i^p + \sum 1 \right]^{\frac{1}{p}} = 0$$

still hold.

Norm inequalities

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty$$

Pf.
The left hand side is easy

$$\|x\|_\infty = \max_{1 \leq j \leq n} \{x_1, x_2, ..., x_n\} = \sqrt{\left(\max_{1 \leq j \leq n} \{x_1, x_2, ..., x_n\} \right)^2} \leq \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

For the right hand side

$$n \|x\|_\infty^2 = n \left(\max_{1 \leq j \leq n} \{x_1, x_2, ..., x_n\} \right)^2 = \sum_{i=1}^{n} \left(\max_{1 \leq j \leq n} \{x_1, x_2, ..., x_n\} \right)^2 \geq \sum_{i=1}^{n} |x_i|^2 = \|x\|_2^2 \geq 0$$

Thus, taking the square-root

$$\sqrt{n} \|x\|_\infty \geq \|x\|_2$$

Combine the inequalities

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty$$

$$-END-$$