Some points about Matrix

Andersen Ang

Firs created: before 2014. Last updated: 2017-Feb-1

Definition 1. A *matrix* is a rectangular array of elements. The elements are member of a $field$. Notation : $A_{m\times n} = \{a_{ij}\}$ with m rows and n columns. a_{ij} is the element in i^{th} row and j^{th} column.

Definition 2. A *square matrix* has same number of rows and columns $A_{n\times n} = \{a_{ij}\}$, with n rows and n columns.

Definition 3. A *vector* is a *matrix* with only one row or only one column. $\mathbf{a}$ denote column vector, $\mathbf{a}^T$ denote row vector.

Definition 4. The *diagonal* of a *square matrix* is the set of all elements a_{ii}.

Definition 5. The *trace* of a *square matrix*, denoted as $\text{Tr}A$, is the sum of all set of all elements a_{ii} : $\text{Tr}\, A_{n\times n} = \sum_{i=1}^{n} a_{ii}$

Definition 6. Matrix equality. $A = B \iff a_{ij} = b_{ij}\ \forall i, j$: Two matrices are equal if 1) they have same size, 2) they have same corresponding elements.

Definition 7. Matrix addition. $A + B = C \iff a_{ij} + b_{ij} = c_{ij}\ \forall i, j$: Two matrix with same size, add the corresponding elements to obtain the result element.

Exercise 8. Matrix addition is *associative* and *commutative* $A + (B + C) = (A + B) + C$, $A + B = B + A$

Definition 9. Matrix multiplication $AB = C \iff \sum_{k=1}^{n} a_{ik}b_{kj}\ \forall i, j$: i^{th} row of A and j^{th} column of B

Exercise 10. Matrix multiplication is *associative* $A(BC) = (AB)C$. But NOT commutative $AB \neq BA$.

Definition 11. Zero matrix , $\mathbf{0} \iff a_{ij} = 0\ \forall i, j$: all elements are zeros

Definition 12. Diagonal matrix is a square matrix with all off-diagonal element as zeros. $D_{n\times n}$, $a_{ij} = 0\,\forall i, j, i \neq j$

Definition 13. Unit matrix is a diagonal matrix with all diagonal element as one. $I_{n\times n}$, $a_{ii} = 1\,\forall i, j$

Exercise 14. $IA = A = AI$

Definition 15. Triangular matrix. A matrix is *upper* triangular matrix if all elements below the diagonal are zeros. The matrix is *lower* triangular matrix if all elements above the diagonal are zeros. Diagonal elements need not to be zero : $a_{ij} = 0\ \forall i \neq j$.

Definition 16. Transpose. For $A_{m\times n} = \{a_{ij}\}$, the *transpose* A^T is a $n \times m$ matrix that $A^T = \{a_{ji}\}$: row becomes column, column becomes row.

Definition 17. Complex conjugate transpose. For $A_{m\times n} = \{a_{ij}\}$, the *complex conjugatetranspose* $A^\dagger$ is a $n\times m$ matrix that $A^\dagger = \{\overline{a_{ji}}\}$, where $\overline{a}$ is the complex conjugate of a.

Exercise 18. $(AB)^T = B^T A^T$ and $(AB)^\dagger = B^\dagger A^\dagger$.

Definition 19. Symetric matrix. A square matrix is *symmtric* if $A = A^T$.

Definition 20. Hermitian matrix. A square matrix is $Hermitian$ if $A = A^\dagger$.

Definition 21. Skew-symetric matrix. A square matrix is $skew - symmtric$ if $A = -A^T$.

Definition 22. Normal matrix. A matrix is *norml* if $A^\dagger A = AA^\dagger$.

Definition 23. Orthogonal matrix. A matrix is *orthogonal* if $A^\dagger A = D$.

Definition 24. Unitary matrix. A matrix is *unitary* if $A^\dagger A = I$.

Definition 25. Determinant. For a matrix A, $\det A = \sum^{n!} (-1)^p\, a_{1p_1} a_{2p_2} ... a_{np_n}$, p_i are permutation of $1, 2, ..., n$

Exercise 26. I is symmetric, skew-symmetric, Hermittian, normal, ortohgonal and unitary, and $\det I = 1$.

Exercise 27. $\det(AB) = \det A \det B$

Definition 28. Elementary operations : (1) interchange two rows (columns), (2) multiply a row (column) by a scalar, (3) adding (2) into another row (column).

Definition 29. Permutation matrix P is a square matrix that all row vectors are unit vector. e.g. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Exercise 30. By $\det(AB) = \det A \det B$, (1) for $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (interchange 2 rows), $\det(AP) = -\det A$ (interchange of two rows change the sign of the determinant), (2) $\det(A \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}) = \det A$, adding a scalar times a row to another row does not change value of determinant.

Definition 31. Minor, $\det M_{ij}$ is the determinant of the matrix after deleting i^{th} row and j^{th} column.

Definition 32. Cofactor $c_{ij} = (-1)^{i+j} \det M_{ij}$

Theorem 33. *Laplace expansion,* $\det A = \sum_{j=1}^{n} a_{ij} c_{ij}$ *for any row i.*

Corollary 34. *Determinant of triangular $n \times n$ matrix is the product of diagonal elements,* $\det T = \prod_i^n a_{ii}$.

Corollary 35. *Determinant of diagonal $n \times n$ matrix is the product of diagonal elements,* $\det D = \prod_i^n a_{ii}$.

Definition 36. Adjugate matrix $adj A = c_{ij}^T$.

Exercise 37. $A \,\mathrm{adj} A = I \det A$. So $A^{-1} = \dfrac{1}{\det A} \mathrm{adj} A$.

Definition 38. The left inverse of $A_{n \times m}$ is B if $BA = I_m$.

Definition 39. The right inverse of $A_{n \times m}$ is C if $AC = I_n$.

Exercise 40. For A having both left inverse and right inverse, $C = IC = BAC = BI = B$. Since $BA = I_m$ and $AC = I_n$, then if $C = B$, A must be square.

Exercise 41. For non-singular matrix A having both left inverse and right inverse, then $C = IC = BAC = BI = B$ and thus nonsingular matrix A must be a square and the inverse is unique, denoted as A^{-1} such that $A^{-1}A = BA = I$ and $AA^{-1} = AC = I$, thus $A^{-1}A = AA^{-1} = I$.

Corollary 42. *By* $\det(AB) = \det A \det B$, $\det(AA^{-1}) = \det A \det A^{-1} = \det I = 1$, *so if* $\det A = 0$, *A have no inverse.*