Some points on linear algebra

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1 Vectors

3 type of quantities : $Scalar$, $Vector$, $Tensor$

$Scalar$: Only magnitude. (Temperature, Speed, Mass ...)

$Vector$: Magnitude + Direction. (Velocity, Force,) $Tensor$: (Permitivity)

1.1 Vector Representation

$\mathbf{v} = \overrightarrow{OP} = \overline{OP} = x\mathrm{i}+y\mathrm{j}+z\mathrm{k} = x\widehat{x}+y\widehat{y}+z\widehat{z} = x\hat{e_1}+y\hat{e_2}+z\hat{e_3} = (x,y,z) = \langle x,y,z\rangle = |OP|\widehat{OP} = \Big|OP\Big\rangle$

Corrdinate System
Magnitude $|\mathbf{v}| = \sqrt{v_1^2+v_2^2}$
Direction $\theta = \tan^{-1}\frac{v_2}{v_1}$
Vector Equality
$\mathbf{v_1} = \mathbf{v_2}$ iff same magnitude **and** same direction
Free vector

1.2 Vector Operation

Position Vector For $A = (a_1, a_2, ..., a_n)$ $B = (b_1, b_2, ..., b_n)$ $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, ..., b_n - a_n)$

Magnitude For vector $\overrightarrow{OP} = (p_1, p_2, ..., p_n)$, magnitude $\left\|\overrightarrow{OP}\right\| = \sqrt{p_1^2 + p_2^2 + ... + p_n^2}$

Negative For $u = -v$, $u = (-v_1, -v_2, ..., -v_n)$. u is anti-parallel to v. They have same magnitude but opposite direction.

Scalar multiplication For $u = kv$, $u = (ku_1, ku_2, ..., ku_n)$. u parallel to v, with magnitude is k times larger.

Addition For $w = u + v$, $w = (u_1 + v_1, u_2 + v_2, ...u_n + v_n)$. The addition obey parallelogram rule.

1.2.1 Dot Product / Inner Product / Scalar Product

Definition. Dot Product is a scalar

$$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3 = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3)$$

$$a \cdot b = |a||b|\text{cos}\theta_{ab} \quad 0 \leq \theta \leq \pi$$

Corollary. *Angle between two vector*

$$cos\theta = \frac{a \cdot b}{|a||b|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}} \quad a, b \neq 0$$

Corollary. *u, v perpendicular to each other* $\Longleftrightarrow u \cdot v = 0$

1.2.2 Cross Product / Outter Product / Vector Product

Definition. Cross Product is a vector that perpendicular to the plane formed by the two vector

$$a \times b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a \times b = |a||b|\text{sin}\theta_{ab}\hat{n}$$

2 Vector space (Linear space)

2.1 Definition of Linear Space

The set V is a linear space if

1. Addition is commutative : $\forall u, v \in V$, $u + v = v + u$
2. Addition is associative : $\forall u, v, w \in V$, $(u + v) + w = u + (v + w) = v + (u + w)$
3. Identity element in addition : $\exists!\ O \in V$ s.t. $\forall u \in V$, $u + O = u$
4. Negative : $\exists!\ v \in V$ s.t. $\forall u \in V$, $u + v = O$
5. Scalar multiplication is communtative, associative, distributive : $\forall a, b \in \mathbb{R}$ and $\forall u, v \in V$, $a(bu) = (ab)u = b(au)$, $(a + b)u = au + bu$, $a(u + v) = au + av$
6. Identity element in scalar multiplication : $\exists!\ 1 \in \mathbb{R}$ s.t. $\forall u \in V$, $1u = u$

2.2 Subspace

For a nonempty subset S of linear space$V$, if S is

1. Close in addition : $\forall u, v \in S$, then $u + v \in S$
2. Close in scalar multiplication : $\forall a, u \in S$, then $au \in S$

Then S is a subspace of V.

2.3 Linear combination and linear dependence

Definition. For two vectors u and v. A linear combination has the form $av + bu$.

If a vector w can be expressed as $w = ax + by$, then w is a linear combination of u and v.

Definition. For a set of vectors $\{v_i\}_{i=1}^{i=n}$, the sum $L = \sum_{i=1}^{i=n} a_i v_i$ where $\{a_i\}_{i=1}^{i=n}$ are scalars, is called a **linear combination** of $\{v_i\}_{i=1}^{i=n}$.

If a vector z can be expressed as $z = \sum_{i=1}^{i=n} a_i v_i$, then z is a linear combination of $\{v_i\}_{i=1}^{i=n}$.

Definition. For a set of vectors$\{v_i\}_{i=1}^{i=n} \in \mathbb{R}^n$, and $\forall x \in \mathbb{R}^n$ can be expressed as linear combination of $\{v_i\}_{i=1}^{i=n}$. Then these vector $\{v_i\}_{i=1}^{i=N}$ **span / generate**$\mathbb{R}^n$ **,** and the set of vector $\{v_i\}_{i=1}^{i=n}$ is called a **basis**.

Definition. Vector set $\{v_i\}_{i=1}^{i=n}$ is **linear independent** iff vector equation $\sum_{i=1}^{i=n} a_i v_i = 0$ has only **trival solution.** If the vector equation has **non-trival solution**, the vector set is **linear dependent**.

For a linear space V with a set of vectors v_1, v_2, ...,v_n. The vectors are linear dependent if there exists a group of non-zero element a_1, a_2, ... , a_n in $\mathbb{R}$ such that

$$\sum_{i=1}^{n} a_i v_i = \mathbf{0}$$

If there is no such set of$\{a_i\}$, then$\{v_i\}$ are all linear independent. Geometrically, it means if all vectors can not be expressed as a linear combination of other vectors, then all vectors are linear independent to each other.

Corollary. *Vectors set $\{v_i\}_{i=1}^{i=n}$ including zero vector must be linearly dependent*

If vector v can be expressed as linear combination of $\{v_i\}_{i=1}^{i=n}$, then vectors set $\{v_i\}_{i=1}^{i=n}$ & v are linearly dependent

If vectors $\{v_i\}_{i=1}^{i=n}$ are linearly dependent, then one of vectors can expressed as linear combination of the other vectors

2.4 Span, basis and dimension

A vector w is *span* by u and v if w can be expressed as a linear combination of u and . Geometrically, that means w can be represented / formed by $au + bv$ for some a, b.

For a set of vectors$\{v_i\}_{i=1}^{i=n}$ in a linear space V , if
1. v_1 , v_2 , ... , v_n are all linearly independent
2. v_1 , v_2 , ... , v_n span V : span $\{v_1, v_2, ..., v_n\} = V$
Then $\{v_i\}_{i=1}^{i=n}$ is a basis for V

For linear space V , if V has a basis of n vectors, then dimension of V is n

3 Matrix

1. For a matrix$A \in \mathbb{R}^{m\times n}$, $A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$

2. Transpose $A^T \in \mathbb{R}^{n\times m}$, $A^T = \begin{bmatrix} a_{11} & a_{21} & ... & a_{m1} \\ a_{12} & a_{22} & ... & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & ... & a_{mn} \end{bmatrix}$

3. Column vector notation $A = [a_{:1}\, a_{:2} ... a_{:n}]$, $A^T = \begin{bmatrix} a_{:1}^T \\ a_{:2}^T \\ \vdots \\ a_{:n}^T \end{bmatrix}$

4. Row vector notation $A = \begin{bmatrix} a_{1:} \\ a_{2:} \\ \vdots \\ a_{m:} \end{bmatrix}$, $A^T = [a_{1:}^T\, a_{2:}^T ... a_{m:}^T]$

5. Matrix-vector multiplication. For $x \in \mathbb{R}^n$, $Ax = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

6. Column space view of matrix-vector multiplication. $Ax = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + ... +$

$$x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = x_1 a_{:1} + x_2 a_{:2} + ... + x_n a_{:n},$$ which means the linear combination of the columns of A with coefficients x.

7. Linear span. For a set of vectors $x_1, \dots, x_n \in \mathbb{R}^k$, the linear span of the set is $\text{span}\left[x_1, ..., x_n\right] = \left\{y \in \mathbb{R}^k | \, y = b_1 x_1 + b_2 x_2 + ... + b_n x_n, \, b_1, ..., b_n \in \mathbb{R}\right\}$

8. Orothogonal subpsace $\text{span}\left[x_1, ..., x_n\right]^\perp = \left\{z \in \mathbb{R}^k | \, z \cdot x_i = 0, \, i = 1, 2, ..., n\right\}$

9. Range of matrix. For $A \in \mathbb{R}^{m \times n}$, range of A is the span of columns of A. $\text{Range}(A) = \text{Span}\left(\{a_{:1}, \, a_{:2} \dots a_{:n}\}\right) = \{y \in \mathbb{R}^m | \exists x \in \mathbb{R}^n \; y = Ax\}$.

10. Row space view of matrix-vector multiplication. $$Ax = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1:} \cdot x \\ a_{2:} \cdot x \\ \vdots \\ a_{m:} \cdot x \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i} x_i \\ \sum_{i=1}^n a_{2i} x_i \\ \vdots \\ \sum_{i=1}^n a_{mi} x_i \end{bmatrix},$$ which means the dot product of x with rows of A.

11. Null Space. For $A \in \mathbb{R}^{m \times n}$, $\text{Null}(A) = \{x \in \mathbb{R}^n | \, Ax = 0\}$. By row space view of $Ax = 0$, $\text{Null}(A)$ = set of x that $\sum_{i=1}^n a_{ji} x_i \; \forall j$. That is, $\text{Null}(A) = \text{Span}\left[a_{1:}, \, a_{2:}, \, ..., a_{m:}\right]^\perp = \text{Rang}\left(A^T\right)^\perp$.

12. Fundamental Theorem of the Alternative. $\text{Null}(A) = \text{Rang}\left(A^T\right)^\perp$ and $\text{Rang}(A) = \text{Null}\left(A^T\right)^\perp$.

13. System of linear equations. For $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The solution set of the problem $Ax = b$ is either empty, a signle point or an infinite set. If a solution $x_0 \in \mathbb{R}^n$ exists, then the set of solution is $x_0 + \text{Nul}(A)$.