1 Matrix Algebra

1.1 Determinant

For a square matrix$A_{n\times n}$, the **determinant of A,** $det\,A$, is a number that is :

$$det\,A = \underbrace{\sum_{i=1}^{n} a_{ij}(-1)^{i+j} det\,M_{ij}}_{\text{Column Expansion}} = \underbrace{\sum_{j=1}^{n} a_{ij}(-1)^{i+j} det\,M_{ij}}_{\text{Row Expansion}}$$

The $\pm$Sign $(-1)^{i+j}$

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The Minor M_{ij} is a $(n-1)\times(n-1)$ submatrix formed by deleting i^{th} row and j^{th} column form $A_{n\times n}$

So for a $n\times n$ determinant, after $n-1$ times of reduction, it reduce into a number.

Example. 2×2

$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}(-1)^{1+1}\det M_{11} + a_{21}(-1)^{2+1}\det M_{21}$

$$= a_{11}a_{22} - a_{21}a_{12}$$

Corollary. *For matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Example. 3×3 , $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$
$= a_{11}(-1)^{1+1}\det M_{11} + a_{12}(-1)^{1+2}\det M_{12} + a_{13}(-1)^{1+3}\det M_{13}$
$= a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$
Apply the 2×2 Corollary ,
$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$
$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

Corollary. *For matrix* $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aef + bfg + cdh - ahf - bdi - gec$$

Or, ***Rule of Sarrus***

1.2 Properties of Determinant

1. Transpose $\det M^T = \det M$
2. Interchange row / column, add a negative sign
3. For any triangular matrix B , $\det B = \text{diag} B$
4. $\det AB = \det A \cdot \det B$ $\quad \left[\ \det \prod_{i=1}^{i=n} A_i \ = \ \prod_{i=1}^{i=n} (\det A_i) \ \right]$

1.3 Application of Determinant

1. **Test existance of inverse.** For square matrix $A_{n\times n}$, if $\det A = 0$, it do not have inverse. i.e. A is a singular matrix

   i.e.

$$\det A = 0 \qquad \Longleftrightarrow \qquad A^{-1} \text{does not exist}$$

$$\Longleftrightarrow \qquad A\overrightarrow{x} = \overrightarrow{0} \quad \text{has non-zero solution}$$

1. **Find the Eigenvalue.** Eigenvalue λ for matrix A can be found by the eqaution : $\det (A_{n\times n} - \lambda I_n) = 0$
2. **Cross Product.** $\overrightarrow{a} \times \overrightarrow{b} = \mathbf{det} \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$

2 Eigenvalue & Eigenvector

Definition. For matrix $A_{n\times n}$, v is a eigenvector of A with eigenvalue λ iff

$$\begin{cases} Av = \lambda v \\ v \neq 0 \end{cases}$$

$(A_{n\times n} - \lambda I_n)\, v = 0$ has non-zero solution iff matrix $(A - \lambda I)$ do not have inverse. i.e. $\det(A - \lambda I) = 0$.

So eigenvalues of A are determined by

$$\det (A - \lambda I) = 0$$

Expansion of such determinant give a **characteristic polynomial** equation in degree n. Has exactly n roots (including complex root, equal roots)

Then with λ, the eigenvector v can be determined by

$$(A - \lambda I)\, v = 0$$

Each eigenvalue has at least one corresponding eigenvector. If eigenvalue λ has multiplicity m , then it may have from 1 to m lineary independent eigenvector

- If A is in order 2 , $\det(A-\lambda I)=\lambda^2-(\mathrm{Tr}A)\,\lambda+\det A$
- If A is ino order 3, $\det(A-\lambda I)=-\lambda^3+(\mathrm{Tr}A)\,\lambda^2-\left(\sum_{i=1}^{i=3}\det M_i\right)\lambda+\det A$
- Hence for triangular $A_{2\times 2}$: $\lambda^2-(\mathrm{Tr}A)\,\lambda=0 \iff \lambda\,(\lambda-\mathrm{Tr}A)=0$
- Hence for triangular $A_{3\times 3}$: $-\lambda^3+(\mathrm{Tr}A)\,\lambda^2-\left(\sum_{i=1}^{i=3}\det M_i\right)\lambda=0 \iff$??
- If A is triangular, then eigenvalue is just the **diagonal entries** of the matrix

Definition. The matrix $A_{n\times n}$, the set of all eigenvectors corresponding to λ, together with zero vector, is a subspace of $\mathbb{R}^n$, the **eigenspace** of λ