Numerical Integrations

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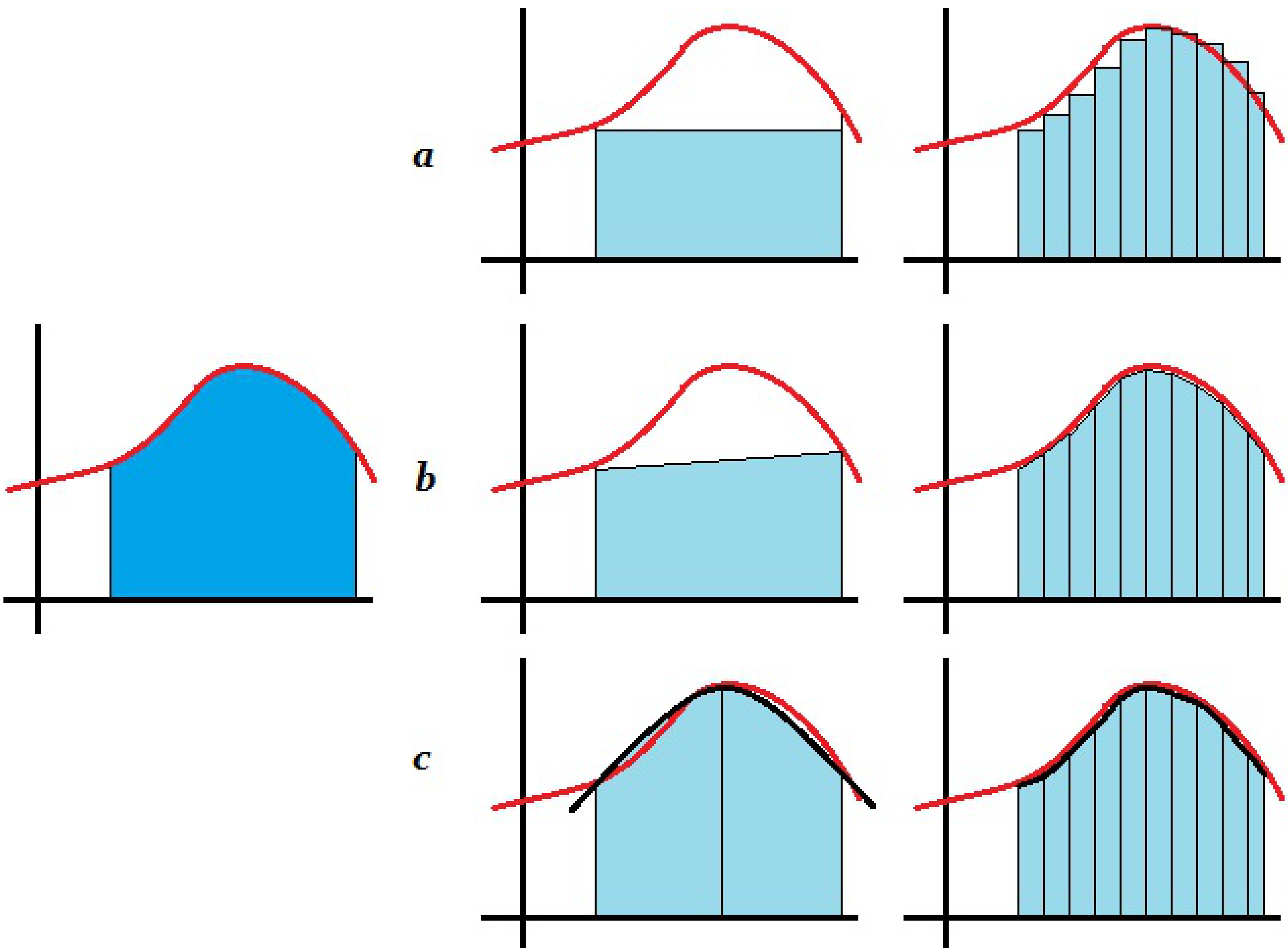
Summary



n-segments rectangles $\int_a^b f(x)dx \approx \sum_{i=1}^{N} f\left(x_i\right) h$

n-segments trapezoidal $\int_a^b f(x)dx \approx \dfrac{h}{2}\Big(f(a) + \sum_{i=1}^{N-1} f\left(a+ih\right) + f(b)\Big)$

n-segments Simpson's $\dfrac{1}{3}$ $\int_a^b f(x)dx \approx \dfrac{h}{3}\left(f(a) + 4\sum_{odd\, i=1}^{n-1} f\left(x_i\right) + 2\sum_{even\, i=1}^{n-2} f(x_i) + f(b)\right)$

1. One-segment rectangle

Using one rectangle to approximate the integral :

$$\int_a^b f(x)dx \approx f(a)(b-a)$$

2. N-segments rectangle

Using step size $x_{k+1} - x_k = h$, starting at $x_0 = a$:

$$\int_a^b f(x)dx \approx f(x_0)h + f(x_1)h + ... + f(x_n)h = \sum_{i=1}^{N} f(x_i)\,h$$

3. One-segment Trapezoid

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left(f(a) + f(b)\right)$$

4. N-segments Trapezoid

$$\int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + ... + \int_{a+(n-1)h}^{a+nh} f(x)dx$$

or using $x_i$notation

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + ... + \int_{x_{n-1}}^{x_n} f(x)dx$$

it is approximated as

$$\approx h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + ... + h\frac{f(x_{n-1}) + f(x_n)}{2}$$

$$= \frac{h}{2}\left[f(x_0) + f(x_1) + f(x_1) + f(x_2) + ... + f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{2}\left[f(x_0) + \sum_{i=1}^{N} f(x_0 + ih) + f(x_n)\right]$$

5. One-segments Simpson $\frac{1}{3}$

Use a quadratic equation to approximate the function.
The quadratic equation is

$$f_{II}(x) = Ax^2 + Bx + C$$

where a, b, c are unknown.

Since there is 3 known, so we need to have 3 points on the curve to provide enough information to find $f_{II}(x)$

$$\begin{cases} (a, f(a)) \\ \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right) \\ (b, f(b)) \end{cases} \implies \begin{cases} Aa^2 + Ba + C = f(a) \\ A\left(\frac{a+b}{2}\right)^2 + B\left(\frac{a+b}{2}\right) + C = f\left(\frac{a+b}{2}\right) \\ Ab^2 + Bb + C = f(b) \end{cases}$$

i.e. In Matrix Form

$$\begin{pmatrix} a^2 & a & 1 \\ \left(\frac{a+b}{2}\right)^2 & \frac{a+b}{2} & 1 \\ b^2 & b & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} f(a) \\ f\left(\frac{a+b}{2}\right) \\ f(b) \end{pmatrix}$$

The solution is

$$A = \frac{b(b+a)f(a) - 4abf\left(\frac{a+b}{2}\right) + a(a+b)f(b)}{(b-a)^2}$$

$$B = \frac{-(3b+a)f(a) + 4(b+a)f\left(\frac{a+b}{2}\right)(b-a)^2 - (3a+b)\,f(b)}{(b-a)^2}$$

$$C = \frac{2f(a) - 4f\left(\frac{a+b}{2}\right) + 2f(b)}{(b-a)^2}$$

So, the integral approximation is

$$\int_a^b f(x)dx \approx \int_a^b f_{II}(x)dx = \int_a^b (Ax^2 + Bx + C)dx = A\frac{b^3 - a^3}{3} + B\frac{b^2 - a^2}{2} + C(b-a)$$

Put in A , B , C, and after simplification

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since $b-a$ is the interval, and there are 3 point, so the step height is $\frac{b-a}{2}$, and thus

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

6. N-segments Simpson

* The value N is an odd number

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + ... + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$\approx \frac{h}{3}\left[f(x_0) + 4f\left(x_1\right) + f(x_2)\right] + ... + \frac{h}{3}\left[f(x_{n-2}) + 4f\left(x_{n-1}\right) + f(x_n)\right]$$

$$= \frac{h}{3}\left(f(x_0) + 4\sum_{odd\, i=1}^{n-1} f(x_i) + 2\sum_{even\, i=1}^{n-2} f(x_i) + f(x_n)\right)$$

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