Lipschitz Constant

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1 What is Lipschitz Constant

First consider a single-variable function $f(x)$ for x inside its domain D.

The magnitude of the slope of $f(x)$ for 2 points $(x_1\,,\,f(x_1))$ and $(x_2\,,\,f(x_2))$ is thus $\left|\dfrac{f(x_1)-f(x_2)}{x_1-x_2}\right|$

A non-negative real number L , which is the smallest upper bound of the slope , (i.e. the "limiter" of the slope) , is called the Lipschitz Constant

$$L = \sup \frac{|f(x_1)-f(x_2)|}{|x_1-x_2|} = \sup \left|\frac{df}{dx}\right|$$

That is, the Lipschitz Constant limit how fast the function can change, in other words, if there is no Lipschitz Constant, the function can change extremely fast without border, that is, the function becomes discontinuous

For 2-variable function

$$L_y = \sup \frac{|f(x,y_1)-f(x,y_2)|}{|y_1-y_2|} = \sup \left|\frac{\partial f}{\partial y}\right|$$

For $n-$variable function

$$L_{x_n} = \sup \left|\frac{\partial f}{\partial x_n}\right|$$

2 Use of Lipschitz Constant in Numerical Analysis

The Cauchy-Lipshitz theorem states:

If $f(x, y)$ is defined in the rectangle R, $x_0 \leqq x \leqq x_0+a$, $y_0-b \leqq y \leqq y_0+b$, continuous in R with respect to x and satisfies the Lipschitz condition

$$\left|f(x, y_1)-f(x, y_2)\right| \leqq A\left|y_1-y_2\right|$$

for all pairs y_1, y_2 uniformly in x, then there exists a uniquely determined function $y(x)$ with $y(x_0)=y_0$ satisfying the differential equation

$$dy/dx = f(x, y) \tag{1}$$

in the interval $x_0 \leqq x \leqq x_0+h$ with $h=\min\,(a, b/M)$ where $M=\max_{(x,y)\in R}\left|f(x, y)\right|$.

3 Computation of Lipschitz Constant

The Lipschitz Constant can be found using definition or triangle-inequality

EXAMPLE
$f = |y|$

$$L = \sup \left| \frac{\partial f}{\partial y} \right| = \sup \left| \frac{\partial y}{\partial y} \right| = \sup 1 = 1$$

EXAMPLE
$f = \sqrt{|y|}$

$$L = \sup \left| \frac{\partial f}{\partial y} \right| = \sup \left| \frac{\partial \sqrt{|y|}}{\partial y} \right| = \sup \frac{1}{\sqrt{|y|}} = \infty \notin \mathbb{R}^+$$

Thus there is no Lipschitz Constant
EXAMPLE
$f = x^2 y$, $x \in [-5, 2]$, $y \in [3, \infty]$

$$L = \sup \left| \frac{\partial f}{\partial y} \right| = \sup \left| x^2 \right| = 25$$

EXAMPLE
$f = \sqrt{x^2 + y^2}$

$$|f(x, y_1) - f(x, y_2)| = |\sqrt{x^2 + y_1^2} - \sqrt{x^2 + y_2^2}|$$

$$= \left| \sqrt{x^2 + y_1^2} - \sqrt{x^2 + y_2^2} \right| \cdot \frac{\sqrt{x^2 + y_1^2} + \sqrt{x^2 + y_2^2}}{\sqrt{x^2 + y_2^2} + \sqrt{x^2 + y_2^2}}$$

$$= \frac{|y_1^2 - y_2^2|}{\sqrt{x^2 + y_2^2} + \sqrt{x^2 + y_2^2}}$$

$$= \frac{|y_1 + y_2|}{\sqrt{x^2 + y_2^2} + \sqrt{x^2 + y_2^2}} |y_1 - y_2|$$

$$\leq \underbrace{\frac{|y_1| + |y_2|}{\sqrt{x^2 + y_2^2} + \sqrt{x^2 + y_2^2}}}_{\leq 1} |y_1 - y_2|$$

$$\leq 1 \cdot |y_1 - y_2|$$

Thus $L = 1$

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