Convergence of different methods for Initial value problem

Forward Euler Method & 2nd RK Method

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$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

Forward Euler Method

The forward Euler Method is

$$u_{n+1} = u_n + hf(x_n, y_n)$$

And the true solution is $y(x_{n+1}) = y_n$ with trancation error τ_n

$$y_{n+1} = y_n + hf(x_n, y_n) + \tau_n$$

Then the error of the method is

$$\epsilon_{n+1} = u_{n+1} - y_{n+1}$$

$$= [u_n + hf(x_n, u_n)] - [y_n + hf(x_n, y_n) + \tau_n]$$

$$= \underbrace{(u_n - y_n)}_{\epsilon_n} + h\left[f(x_n, u_n) - f(x_n, y_n)\right] - \tau_n$$

Take absolute value ($\Delta inequality: \; a = b + c \Rightarrow |a| \leq |b| + |c|$)

$$|\epsilon_{n+1}| \leq |\epsilon_n| + h\left|f(x_n, u_n) - f(x_n, y_n)\right| + |\tau_n|$$

Apply Lipschitz

$$|\epsilon_{n+1}| \leq |\epsilon_n| + hL\underbrace{|u_n - y_n|}_{\epsilon_n} + |\tau_n|$$

$$|\epsilon_{n+1}| \leq (1 + Lh)|\epsilon_n| + |\tau_n|$$

Assume

$$\tau = \sup|\tau_n|$$

Thus

$$|\epsilon_{n+1}| \leq (1 + Lh)|\epsilon_n| + \tau$$

$$|\epsilon_{n+1}| \leq (1 + Lh)\left((1 + Lh)\left|\epsilon_{n-1}\right| + \tau\right) + \tau$$

$$= (1+Lh)^2|\epsilon_{n-1}| + (1+(1+Lh))\,\tau$$

$$\leq (1+Lh)^3|\epsilon_{n-2}| + \left(1+(1+Lh)+(1+Lh)^2\right)\tau$$

$$\vdots$$

$$\leq (1+Lh)^n|\epsilon_1| + \left(1+(1+Lh)+...+(1+Lh)^{n-1}\right)\tau$$

$$\leq (1+Lh)^{n+1}|\epsilon_0| + \underbrace{\left(1+(1+Lh)+...+(1+Lh)^{n}\right)}_{G.S.}\tau$$

$$|\epsilon_{n+1}| \leq (1+Lh)^{n+1}|\epsilon_0| + \frac{1-(1+Lh)^{n+1}}{1-(1+Lh)}\tau$$

Now consider

$$1+Lh = 1+Lh$$

$$\leq 1+Lh+\frac{(Lh)^2}{2}+\frac{(Lh)^3}{3}+... = e^{Lh}$$

Thus

$$1+Lh \leq e^{Lh}$$

Since $L, h > 0$ (Recall $L = \sup\left|\frac{\partial f}{\partial y}\right|$, and h = step size > 0)

$$(1+Lh)^n \leq e^{nLh} \qquad \forall n = 1,2,...$$

Apply this into the previous inequality

$$|\epsilon_{n+1}| \leq (1+Lh)^{n+1}|\epsilon_0| + \frac{1-(1+Lh)^{n+1}}{1-(1+Lh)}\tau \quad \Longrightarrow |\epsilon_{n+1}| \leq e^{(n+1)Lh}|\epsilon_0| + \frac{1-e^{(n+1)Lh}}{-Lh}\tau$$

i.e.

$$|\epsilon_{n+1}| \leq e^{(n+1)Lh}|\epsilon_0| + \frac{e^{(n+1)Lh}-1}{Lh}\tau$$

Now consider n instead of $n+1$

$$|\epsilon_n| \leq e^{nLh}|\epsilon_0| + \frac{e^{nLh}-1}{Lh}\tau$$

Let $b = nh$

$$|\epsilon_n| \leq \underbrace{e^{Lb}|\epsilon_0|}_{\text{Initial condition error}} + \underbrace{\frac{e^{Lb}-1}{Lb}n\tau}_{\text{Numerical method error}}$$

If initial condition error is zero

$$|\epsilon_n| \leq \frac{e^{Lb}-1}{Lb}n\tau$$

A numerical method is convergent if

$$\lim_{n\to\infty} |\epsilon_n| = 0$$

2^{nd} order Runge-Kutta Method

The 2^{nd} order Runge-Kutta Method (Trapezoidal Method) is in the form

$$\text{Approximation : } u_{n+1} = u_n + \frac{h}{2}\left[f(x_n, u_n) + f(x_{n+1}, u_{n+1})\right] \quad \text{True : } y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right] + \tau_n$$

$$\epsilon_{n+1} = u_{n+1} - y_{n+1}$$

$$= \left[u_n + \frac{h}{2}\left[f(x_n, u_n) + f(x_{n+1}, u_{n+1})\right]\right] - \left[y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right] + \tau_n\right]$$

$$= \underbrace{(u_n - y_n)}_{\epsilon_n} + \frac{h}{2}\left(f(x_n, u_n) - f(x_n, y_n) + f(x_{n+1}, u_{n+1}) - f(x_{n+1}, y_{n+1})\right) - \tau_n$$

Take absolute value, then apply Lipschitz

$$|\epsilon_{n+1}| \leq |\epsilon_n| + \frac{h}{2}\left[\underbrace{|f(x_n, u_n) - f(x_n, y_n)|}_{Lipschitz} + \underbrace{|f(x_{n+1}, u_{n+1}) - f(x_{n+1}, y_{n+1})|}_{Lipschitz}\right] + |\tau_n|$$

Now consider $\left|f(x_n, u_n) - f(x_n, y_n)\right|$, by Lipschitz,

$$\left|f(x_n, u_n) - f(x_n, y_n)\right| \leq L\left|u_n - y_n\right| = L|\epsilon_n|$$

Thus the previous inequality becomes

$$|\epsilon_{n+1}| \leq |\epsilon_n| + \frac{h}{2}\left[L|\epsilon_n| + L\left|u_{n+1} - y_{n+1}\right|\right] + |\tau_n|$$

$$= |\epsilon_n| + \frac{h}{2}\left[L|\epsilon_n| + L\underbrace{\left|u_n - y_n + hf\left(x_n, u_n\right) - hf\left(x_n, y_n\right)\right|}_{\text{Apply}\triangle\text{inequality}}\right] + |\tau_n|$$

$$\leq |\epsilon_n| + \frac{h}{2}\left[L|\epsilon_n| + L\underbrace{\left|u_n - y_n\right|}_{\epsilon_n} + h\left|f\left(x_n, u_n\right) - f\left(x_n, y_n\right)\right|\right] + |\tau_n|$$

Apply Lipschitz again , thus

$$|\epsilon_{n+1}| \leq |\epsilon_n| + \frac{h}{2}\left[2L|\epsilon_n| + hL\left|\epsilon_n\right|\right] + |\tau_n|$$

$$|\epsilon_{n+1}| \le \left(1 + hL + \frac{h^2L^2}{2}\right)|\epsilon_n| + |\tau_n|$$

Now let
$$\tau = \sup|\tau_n|$$

$$|\epsilon_{n+1}| \le \left(1 + hL + \frac{h^2L^2}{2}\right)|\epsilon_n| + \tau$$

$$|\epsilon_{n+1}| \le \left(1 + hL + \frac{h^2L^2}{2}\right)\left[\left(1 + hL + \frac{h^2L^2}{2}\right)|\epsilon_{n-1}| + \tau\right] + \tau$$

$$= \left(1 + hL + \frac{h^2L^2}{2}\right)^2|\epsilon_{n-1}| + \left[1 + \left(1 + hL + \frac{h^2L^2}{2}\right)\right]\tau$$

$$\le \left(1 + hL + \frac{h^2L^2}{2}\right)^3|\epsilon_{n-2}| + \left[1 + \left(1 + hL + \frac{h^2L^2}{2}\right) + \left(1 + hL + \frac{h^2L^2}{2}\right)^2\right]\tau$$

$$\vdots$$

$$\le \left(1 + hL + \frac{h^2L^2}{2}\right)^n|\epsilon_1| + \left[1 + \left(1 + hL + \frac{h^2L^2}{2}\right) + ... + \left(1 + hL + \frac{h^2L^2}{2}\right)^{n-1}\right]\tau$$

$$|\epsilon_{n+1}| \le \left(1 + hL + \frac{h^2L^2}{2}\right)^{n+1}|\epsilon_0| + \underbrace{\left[1 + \left(1 + hL + \frac{h^2L^2}{2}\right) + ... + \left(1 + hL + \frac{h^2L^2}{2}\right)^n\right]}_{G.S.}\tau$$

Thus
$$|\epsilon_{n+1}| \le \left(1 + hL + \frac{h^2L^2}{2}\right)^{n+1}|\epsilon_0| + \frac{1 - \left(1 + hL + \frac{h^2L^2}{2}\right)^{n+1}}{1 - \left(1 + hL + \frac{h^2L^2}{2}\right)}\tau$$

Now consider an equality (this inequality is for simplifying the expression)

$$\left(1 + hL + \frac{h^2L^2}{2}\right) \le 1 + hL + \frac{h^2L^2}{2} + \frac{h^3L^3}{3!} + ... = e^{Lh}$$

Thus, consider n case (not consider $n+1$ to simplify the expression)

$$|\epsilon_n| \le \left(1 + hL + \frac{h^2L^2}{2}\right)^n|\epsilon_0| + \frac{1 - \left(1 + hL + \frac{h^2L^2}{2}\right)^n}{1 - \left(1 + hL + \frac{h^2L^2}{2}\right)}\tau$$

$$|\epsilon_n| \le \underbrace{e^{nLh}|\epsilon_0|}_{\text{Initial condition error}} + \underbrace{\frac{e^{nLh} - 1}{hL + \frac{h^2L^2}{2}}\tau}_{\text{Numerical method error}}$$

If the initial condition error is zero

$$|\epsilon_n| \le \frac{e^{nLh} - 1}{hL + \frac{h^2L^2}{2}}\tau \le \frac{e^{nLh} - 1}{hL}\tau$$

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