Finite Difference Method

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1. FDM on 2nd order ODE

The ODE	Range	B.C.
$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=f(x)$	$x\in(a,b)$	$y(a)=\alpha$ $y(b)=\beta$

Step 1. Convert the Differential Eqaution into Difference Equation

With step size $h=\frac{b-a}{N}$ and thus $\Delta x=x_k-x_{k-1}$ or $x_k=x_{k-1}+h=x_0+kh$

To approximate $\frac{dy}{dx}\approx\frac{\Delta y}{\Delta x}$, there are 3 ways

1. Forward Difference $\frac{dy}{dx}\approx\frac{y_{n+1}-y_n}{h}$
2. Backward Difference $\frac{dy}{dx}\approx\frac{y_n-y_{n-1}}{h}$
3. Central Difference $\frac{dy}{dx}\approx\frac{y_{n+1}-y_{n-1}}{2h}$

Then, to approximate $\frac{d^2y}{dx^2}$, apply foward-backward difference

1st approximation , backward difference $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)\approx\frac{y'_n-y'_{n-1}}{h}$

2nd approximation , forward difference

$$\frac{d^2y}{dx^2}\approx\frac{\left(\frac{dy}{dx}\right)_n-\left(\frac{dy}{dx}\right)_{n-1}}{h}\approx\frac{\frac{y_{n+1}-y_n}{h}-\frac{y_n-y_{n-1}}{h}}{h}=\frac{y_{n+1}-2y_n+y_{n+1}}{h^2}$$

Thus, for central difference on y' and forward-backward difference on y"

$$y'=\frac{dy}{dx}\approx\frac{y_{n+1}-y_{n-1}}{2h}\qquad y"=\frac{d^2y}{dx^2}\approx\frac{y_{n+1}-2y_n+y_{n-1}}{h^2}$$

Plug this approximation into the 2nd order ODE

$$\frac{y_{n+1}-2y_n+y_{n-1}}{h^2}+p_n\frac{y_{n+1}-y_{n-1}}{2h}+q_ny_n=f_n$$

where p_n , q_n , f_n are the approximations of $p(x_n)$, $q(x_n)$, $f(x_n)$

Multiply the whole equation by h^2 , and group same terms, the difference equation is thus

$$y_{n-1}\left[1-\frac{h}{2}p_n\right]+y_n\left[-2+h^2q_n\right]+y_{n+1}\left[1+\frac{h}{2}p_n\right]=h^2f_n$$

Step 2. Iteration

We can now apply the difference equation again again and again

Recall $n = 1, 2, ..., N-1$, because for $n = N-1$, $n+1 = N$ which is the number of step

Let $a_n = -2 + h^2 q_n$, $b_n = 1 - \frac{h}{2} p_n$, $c_n = 1 + \frac{h}{2} p_n$

$$\begin{cases} y_{1-1}\left[1 - \frac{h}{2}p_1\right] + y_1\left[-2 + h^2 q_1\right] + y_{1+1}\left[1 + \frac{h}{2}p_1\right] = h^2 f_1 \\ y_{2-1}\left[1 - \frac{h}{2}p_2\right] + y_2\left[-2 + h^2 q_2\right] + y_{2+1}\left[1 + \frac{h}{2}p_2\right] = h^2 f_2 \\ y_{3-1}\left[1 - \frac{h}{2}p_3\right] + y_3\left[-2 + h^2 q_3\right] + y_{3+1}\left[1 + \frac{h}{2}p_3\right] = h^2 f_3 \\ \vdots \\ y_{(n-1)-1}\left[1 - \frac{h}{2}p_{n-1}\right] + y_{n-1}\left[-2 + h^2 q_{n-1}\right] + y_{(n-1)+1}\left[1 + \frac{h}{2}p_{n-1}\right] = h^2 f_{n-1} \end{cases}$$

Become

$$\begin{array}{lllllll} b_1 y_0 & +a_1 y_1 & +c_1 y_2 & & & & = h^2 f_1 \\ & +b_2 y_1 & +a_2 y_2 & +c_2 y_3 & & & = h^2 f_2 \\ & & +b_3 y_2 & +a_3 y_3 & +c_3 y_4 & & = h^2 f_3 \\ & & & \vdots & & & \\ & & & +b_{n-1} y_{n-2} & +a_{n-1} y_{n-1} & +c_{n-1} y_n & = h^2 f_{n-1} \end{array}$$

By the boundary condition $y(a) = \alpha$, $y(b) = \beta$

$$\begin{array}{lllllll} b_1 \alpha & +a_1 y_1 & +c_1 y_2 & & & & = h^2 f_1 \\ & +b_2 y_1 & +a_2 y_2 & +c_2 y_3 & & & = h^2 f_2 \\ & & +b_3 y_2 & +a_3 y_3 & +c_3 y_4 & & = h^2 f_3 \\ & & & \vdots & & & \\ & & & +b_{n-1} y_{n-2} & +a_{n-1} y_{n-1} & +c_{n-1} \beta & = h^2 f_{n-1} \end{array}$$

Can be rearrange to

$$\begin{array}{llllll} +a_1 y_1 & +c_1 y_2 & & & & = h^2 f_1 - b_1 \alpha \\ +b_2 y_1 & +a_2 y_2 & +c_2 y_3 & & & = h^2 f_2 \\ & +b_3 y_2 & +a_3 y_3 & +c_3 y_4 & & = h^2 f_3 \\ & & \vdots & & & \\ & & & +b_{n-1} y_{n-2} & +a_{n-1} y_{n-1} & = h^2 f_{n-1} - c_{n-1} \beta \end{array}$$

In Matrix form

$$\underbrace{\begin{bmatrix} a_1 & c_1 & & & & \\ b_2 & a_2 & c_2 & & & \\ & b_3 & a_3 & c_3 & & \\ & & & \ddots & & \\ & & & & b_{n-1} & a_{n-1} \end{bmatrix}}_{(n-1)\times(n-1)} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}}_{(n-1)\times 1} = \underbrace{\begin{bmatrix} h^2 f_1 - b_1 \alpha \\ h^2 f_2 \\ h^2 f_3 \\ \vdots \\ h^2 f_{n-1} - c_{n-1}\beta \end{bmatrix}}_{(n-1)\times 1}$$

with

$$\begin{bmatrix} a_n = -2 - h^2 q_n \\ b_n = 1 - \frac{h}{2} p_n \\ c_n = 1 + \frac{h}{2} p_n \end{bmatrix}$$

2. Example

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 2x \qquad y(0) = 2 \; y(1) = e^{-3} \; x \in [0,1] \text{ find } y(0.5)$$

First ,let find the **analytic solution**
(this is very very very very very standard stuff in year one maths !)

First consider $y" + 5y' + 6y = 0$
Plug in $y = Ae^{\lambda x}$, the equation becomes $A\lambda^2 e^{\lambda x} + 5A\lambda e^{\lambda x} + 6Ae^{\lambda x} = 0$
The characteristic equation is thus $\lambda^2 + 5\lambda + 6 = 0$, or $(\lambda + 2)(\lambda + 3) = 0$ with roots $-2, -3$
Thus $y = Ae^{-2x} + Be^{-3x}$

Then to solve the inhomogenous solution, apply the variation of parameter with wronskian

$$W = \det \begin{pmatrix} e^{-2x} & e^{-3x} \\ \frac{d}{dx}e^{-2x} & \frac{d}{dx}e^{-3x} \end{pmatrix} = -e^{-5x}$$

$$u_1 = -\int \frac{e^{-3x} 2x}{W} dx = \frac{e^{2x} 2x - 1}{2}$$

$$u_2 = \int \frac{e^{-2x} 2x}{W} dx = -\frac{2}{9} e^{3x} (3x - 1)$$

Thus finally the complete solution is thus

$$y = Ae^{-2x} + Be^{-3x} + e^{-2x} \left(\frac{e^{2x} 2x - 1}{2} \right) + e^{-3x} \left[-\frac{2}{9} e^{3x} (3x - 1) \right]$$

$$y = Ae^{-2x} + Be^{-3x} + \frac{x}{3} - \frac{5}{18}$$

With $y(0) = 2$ and $y(1) = e^{-3}$, $A = 6.4031$, $B = -4.1253$

$$y = 6.4031e^{-2x} - 4.1253e^{-3x} + \frac{x}{3} - \frac{5}{18}$$

Thus $y(0.5)$

$$y(0.5) = 1.324$$

Then, let's apply **FDM** with step size 0.1, thus there are $\dfrac{1-0}{0.1} = 10$ steps

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 2x \quad y(0) = 2 \quad y(1) = e^{-3}$$

$$p_n = 5 \qquad q_n = 6 \qquad f_n = 2x_n = 0.2n$$

$$a_n = -2 - h^2 q_n = -2 - 0.01 \cdot 6 = -2.06$$

$$b_n = 1 - \frac{h}{2}p_n = 1 - \frac{0.1}{2}5 = 0.75$$

$$c_n = 1 + \frac{h}{2}p_n = 1 + \frac{0.1}{2}5 = 1.25$$

$$h^2 f_n = 0.002n$$

$$\begin{bmatrix} -2.06 & 1.25 & & & & & & & \\ 0.75 & -2.06 & 1.25 & & & & & & \\ & 0.75 & -2.06 & 1.25 & & & & & \\ & & 0.75 & -2.06 & 1.25 & & & & \\ & & & 0.75 & -2.06 & 1.25 & & & \\ & & & & 0.75 & -2.06 & 1.25 & & \\ & & & & & 0.75 & -2.06 & 1.25 & \\ & & & & & & 0.75 & -2.06 & 1.25 \\ & & & & & & & 0.75 & -2.06 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ [\mathbf{y_5}] \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{bmatrix} = \begin{bmatrix} 0.002 - 0.75 \cdot 2 \\ 0.004 \\ 0.006 \\ 0.008 \\ 0.010 \\ 0.012 \\ 0.014 \\ 0.016 \\ 0.018 - 1.25e^{-3} \end{bmatrix}$$

Which is

$$\begin{bmatrix} -2.06 & 1.25 & & & & & & & \\ 0.75 & -2.06 & 1.25 & & & & & & \\ & 0.75 & -2.06 & 1.25 & & & & & \\ & & 0.75 & -2.06 & 1.25 & & & & \\ & & & 0.75 & -2.06 & 1.25 & & & \\ & & & & 0.75 & -2.06 & 1.25 & & \\ & & & & & 0.75 & -2.06 & 1.25 & \\ & & & & & & 0.75 & -2.06 & 1.25 \\ & & & & & & & 0.75 & -2.06 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ [\mathbf{y_5}] \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{bmatrix} = \begin{bmatrix} -1.498 \\ 0.004 \\ 0.006 \\ 0.008 \\ 0.010 \\ 0.012 \\ 0.014 \\ 0.016 \\ -0.0442 \end{bmatrix}$$

The required solution $y(0.5) = y_5$

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