Optimization Terminology

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This passage discuss some terms in optimization theory.

1. The Problem

$$\text{Optimize } f(x) \text{ subject to } c_i(x)$$

For max problem

$$\max_{x\in\mathbb{R}^n} f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \text{ subject to } \begin{cases} c_i\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = 0 \; i \in E \\ c_i\begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix} \geq 0 \; i \in I \end{cases}$$

It equivalent to min problem

$$\min_{x\in\mathbb{R}^n} \; - f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \text{ subject to } \begin{cases} c_i\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = 0 \; i \in E \\ c_i\begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix} \geq 0 \; i \in I \end{cases}$$

- The $f(.)$ stand for the *objective function* or the *cost function*
- The x , or in explicit form $x = [x_1 \; x_2 \; ... \; x_n]^T$ is the *unknown variable vector* , normally the vector consist of *more than one* vairable.
- The $c_i(.)$ stand for the constraint function (usually more than one constraint $i > 1$)
- The constraints can be a equality constraints $i \in E$ or inequality constraints $i \in I$
- The constraint function can be a function that the whole vector is being involved or just some part of it.
- Maximize the $f(.)$ is equivalent to minimize $-f(.)$
- Sometimes transformations need to be done to transform the problem into these standard form.

2. Optimization of unconstrainted problem is actually the L_∞norm

Since optimization problem is to find the max value of a function within the constraints, thus for problem that $c_i(.) = 0$, i.e. no constraints. The problem becomes

$$\max_{x\in[a,b]} f(x)$$

Then , recall that this is actually the L_∞ norm (for continuous function f)

$$\max_{x\in[a,b]} f(x) = \sup f(x) = ||f(x)||_\infty = \lim_{p\to\infty} \left(\int_a^b |f(x)|^\infty dx \right)^{\frac{1}{p}}$$

3. The max and arg max

Sometimes, optimization problem is not only to find the max value of the cost function f , the information that "what value of x" can maximize f is also under concern. In this case, the problem becomes

$$\text{Find } x* \text{s.t. } x* \ \max \ f(x)$$

Or using the notation arg max

$$x* = \arg\max \ f(x)$$

4. Convex Combination

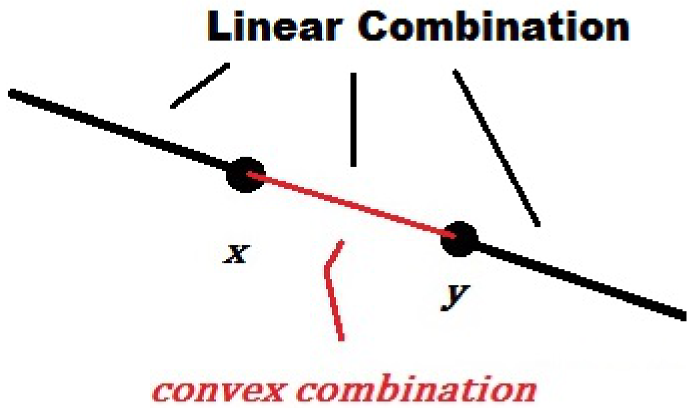
Like *linear combination* , if a point z is the *convex combination* of point x and y

$$z = ax + (1-a)y \qquad a \in [0,1]$$

- Notice that $a \in [0,1]$, that means $0 \le a \le 1$, what this means is that a is a ratio, and the whole expression for z is a *weighted sum* of x, y.
- For large a , then z is closer to x (it "look" like x), for a small a , z is closer to y (it "looks" like y)
- Then this z simply means all possible points that is generated by point x and y using this weighting.
- The more general convex combination is

$$z = a_1x_1 + a_2x_2 + ... + a_nx_n = \sum_{i=1}^{n} a_ix_i \qquad \text{where } \sum_{i=1}^{n} a_i = 1$$

- Linear Combination and Convex Combination



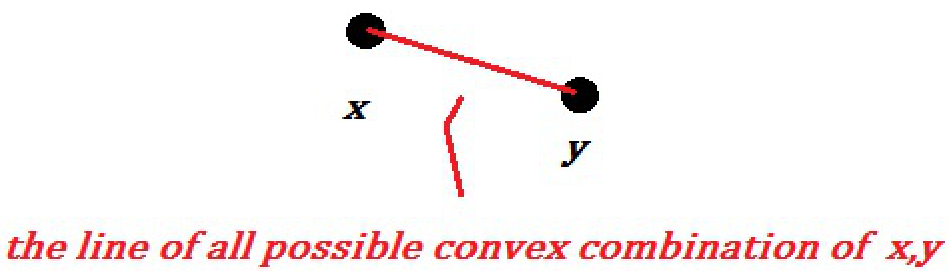
- Thus, it can be seen that convex combination is a special kind of linear combination.

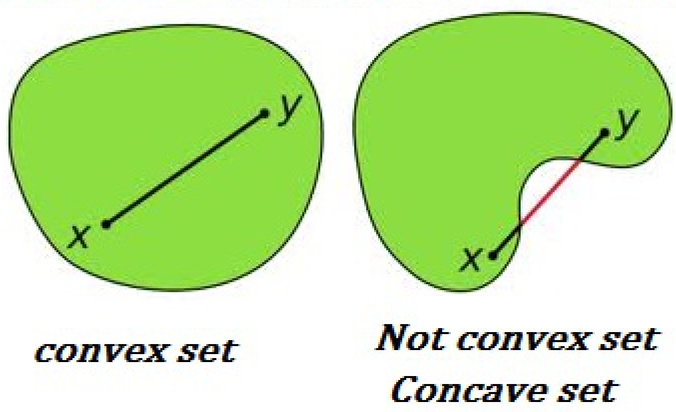
5. **Convexity**

A set S is convex if

$$\forall x, y \in S\,,\ \text{the point}\ z = ax + (1-a)y \in S\ \forall a \in [0,1]$$

- Or in other words, a set S is convex if for any 2 points inside the S , all possible points that span by this 2 points are also inside the S, then S is convex.
- Geometrically, the set is "convex" too



- 

A function $f : A \to B$ is convex if domain A is convex and

$$\forall x, y \in A\,,\ f(ax + (1-a)y) \le af(x) + (1-a)f(y)$$

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