Unconstrained Optimization

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First-Order Condition

Suppose $x*$ is a local maximizer of f , and suppose f is differentiable at $x*$, then

$$Df(x*) = 0$$

Second-Order Condition

If f has a local maximum at $x*$, then $D^2 f(x*)$ is negative semidefinite

Definiteness

Recall that a matrix is positive semidefinite if all eigenvalues≥ 0.

Review on Taylor's Expansion

Univariate $$f(x+\delta) = f(x) + f'(x)\delta + \frac{1}{2}f"(x)\delta^2 + ...$$

Bi-variate $$f(x_1+\delta_1, x_2+\delta_2) = f(x_1, x_2) + \frac{\partial f}{\partial x_1}|_{(x_1,x_2)}\delta_1 + \frac{\partial f}{\partial x_2}|_{(x_1,x_2)}\delta_2 + ...$$

Multi-variate $$f(\mathbf{x}+\delta) = f(\mathbf{x}) + Df(\mathbf{x})^T \cdot \delta + \frac{1}{2}\delta^T \cdot D^2 f(\mathbf{x}) \cdot \delta + ...$$

Example $f = 2x^3 - 3x^2$

$$\frac{df}{dx} = 6x^2 - 6x \qquad \frac{d^2 f}{dx^2} = 12x - 6$$

To find critical point of f , solve $\frac{df}{dx} = 0 : 6x^2 - 6x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 1 \; or \; 0$

So is this critical point a local optimizer of f ? Check the $\frac{d^2 f}{dx^2}$: $f"(0) = -6 < 0$ and $f"(1) = +6 > 0$

Thus 0 is a local maximizer of f and 1 is a local minimizer of f

Example $f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$

$$\nabla f = \begin{bmatrix} 200(x_2 - x_1^2)(-2x_1) + 2(1-x_1)(-1) \\ 200(x_2 - x_1^2) \end{bmatrix} = \begin{bmatrix} 400(x_1^3 - x_2 x_1) + 2(x_1 - 1) \\ 200(x_2 - x_1^2) \end{bmatrix}$$

$$\nabla^2 f = \begin{bmatrix} 400(3x_1^2 - x_2) + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}$$

To find critical point of f , solve $\nabla f = 0$

$$\begin{bmatrix} 400(x_1^3 - x_2 x_1) + 2(x_1 - 1) \\ 200(x_2 - x_1^2) \end{bmatrix} = 0 \Rightarrow \begin{cases} 400x_1^3 - 400x_1 x_2 + 2x_1 - 2 = 0 \\ 200(x_2 - x_1^2) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 200x_1^3 - 200x_1 x_2 + x_1 - 1 = 0 \\ x_2 = x_1^2 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 1 \end{cases}$$

So is this critical point a local optimizer of f ? Check the matrix definiteness

$$\nabla^2 f|_{(1,1)} = \begin{bmatrix} 400(2) + 2 & -400 \\ -400 & 200 \end{bmatrix} = \begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix}$$

$$\lambda\left\{\nabla^2 f|_{(1,1)}\right\} : \det\left(\nabla^2 f|_{(1,1)} - \lambda I\right) = \begin{vmatrix} 802 - \lambda & -400 \\ -400 & 200 - \lambda \end{vmatrix} = 400 - 1002\lambda + \lambda^2$$

$$\lambda = 0.4 \; or \; 1001.6 > 0$$

So $\nabla^2 f|_{(1,1)} > 0$, it is positive definite at $(1, 1)$.

Thus $(1, 1)$ is a local minimizer of f, and value is $f(1, 1) = 0$

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